Introduction to Symplectic Geometry

(Early Draft)

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Part I

Foundations

Chapter 1

Hamiltonian Functions

The aim of this chapter is to quickly acquaint the reader with the most basic elements of symplectic geometry. These ideas are among the most important in the course.

Key Points:

1. A symplectic manifold (M, ω) is a smooth manifold M equipped with a closed and nondegenerate 2-form ω.
2. Every smooth function $f \in C^\infty(M)$ induces a vector field $X_f \in \mathfrak{X}(M)$ which preserves the symplectic structure ω.
3. The interaction between functions and symmetries endows $C^\infty(M)$ with the structure of a Lie algebra.
4. The symplectic sphere $(S^2, \mathrm{d}\theta \wedge \mathrm{d}h)$ is an example of nearly every concept in this course.

Remark. Be advised when consulting the literature: Some sources present definitions which differ by a factor of -1 from our own.

1.1 The Category of Symplectic Manifolds

Definition 1. A *symplectic manifold* is a smooth manifold M equipped with a closed and nondegenerate 2-form $\omega \in \Omega^2(M)$. By *nondegenerate*, we mean that the interior product $\iota_X \omega$ is nonzero for all nonzero $X \in TM$.

Definition 2. Let (M, ω) and (M', ω') be two manifolds. A function $\phi : M \to M'$ is called a *symplectic map* if $\omega = \phi^* \omega'$. If ϕ is additionally a diffeomorphism, then we say that ϕ is a *symplectomorphism* between (M, ω) and (M', ω').

category:	*smooth*	*symplectic*
	smooth map	symplectic map
	diffeomorphism	symplectomorphism

Definition 3. We say that $X \in \mathfrak{X}(M)$ is a *symplectic vector field* if $\mathcal{L}_X \omega = 0$.

If the symplectic vector field $X \in \mathfrak{X}(M)$ is integrable, then for the time-t flow $\phi_t : M \to M$ of X is a symplectic transformation of (M, ω).

1.2 Hamiltonian Functions and Hamiltonian Vector Fields

The nondegeneracy of ω implies that, for each $x \in M$, the map

$$\begin{aligned} T_x M &\xrightarrow{\sim} T_x^* M \\ X &\longmapsto \iota_X \omega \end{aligned}$$

is a linear isomorphism. Extending this map to every fiber of $TM \to M$ yields a linear isomorphism

$$\begin{aligned} \mathfrak{X}(M) &\xrightarrow{\sim} \Omega^1(M) \\ X &\longmapsto \iota_X \omega \end{aligned}$$

Definition 4. The *Hamiltonian vector field* $X \in \mathfrak{X}(M)$ associated to a function $f \in C^\infty(M)$ is defined by

$$\iota_X \omega = \mathrm{d}f.$$

In this case, we also say that f is a *Hamiltonian function* associated to X.

We will frequently denote the Hamiltonian function associated to $f \in C^\infty(M)$ by X_f. Another common notation is H_f.

Note that

$$\begin{array}{ccccc} C^\infty(M) & \xrightarrow{\mathrm{d}} & \Omega^1(M) & \xrightarrow{\sim} & \mathfrak{X}(M) \\ f & & \mathrm{d}f & & X_f \end{array}$$

where the isomorphism is induced by ω.

The most important property of Hamiltonian vector fields, for our purposes in these notes, is the following:

Proposition 5. *If $X \in \mathfrak{X}(M)$ is a Hamiltonian vector field, then X is symplectic.*

Proof. Let $f \in C^\infty(M)$ be a Hamiltonian function for X, so that $\iota_X \omega = \mathrm{d}f$. Using the Cartan homotopy formula[1] and the closedness of ω, we deduce that

$$\begin{aligned} \mathcal{L}_X \omega &= \mathrm{d}\iota_X \omega + \iota_X \mathrm{d}\omega \\ &= \mathrm{d}(\mathrm{d}f) + 0 \\ &= 0. \end{aligned}$$

□

This easy proof establishes the following fundamental property of symplectic manifolds:

The smooth functions on M describe infinitesimal symmetries of (M, ω).

Remark. We may compare this situation to the case of gradient vector fields on a Riemannian manifold (M, g). In the Riemannian case, a function $f \in C^\infty(M)$ describes a gradient vector field ∇f which is *not* in general an infinitesimal transformation of (M, g). That is, it is not always the case that ∇f preserves the metric g.

The formalism of Hamiltonian functions and Hamiltonian vector fields endows the space of functions $C^\infty(M)$ with a special structure, defined as follows.

Definition 6. Let (M, ω) be a symplectic manifold. Define the *Poisson bracket associated to ω*,

$$\{\,,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

by

$$\{f, h\} = X_f h$$

for all $f, h \in C^\infty(M)$.

[1] namely, that $\mathcal{L}_X = \mathrm{d}\iota_X + \iota_X \mathrm{d}$.

That is, $\{f, h\} \in C^\infty(M)$ measures the infinitesimal rate of change of $h \in C^\infty(M)$ along the infinitesimal symmetry $X_h \in \mathfrak{X}(M)$ associated to $f \in C^\infty(M)$. Equivalently, we have

$$\begin{aligned}\{f, h\} &= X_f h \\ &= \mathrm{d}h(X_f) \\ &= \omega(X_h, X_f) \\ &= -\omega(X_f, X_h).\end{aligned}$$

In the exercises, we will show that $\{\,,\}$ is a Lie bracket and a bi-derivation on $C^\infty(M)$. This motivates the following definition.

Definition 7. Let M be a smooth manifold. A *Poisson bracket* on M is a Lie bracket $\{\,,\}$ on $C^\infty(M)$ such that

$$\{f, hk\} = \{f, h\}k + h\{f, k\}$$

for all $f, h, k \in C^\infty(M)$. That is, $\{\,,\}$ is a Lie bracket and a *bi-derivation* on $C^\infty(M)$. The pair $(M, \{\,,\})$ is called a *Poisson manifold*.

Thus, every symplectic manifold is naturally a Poisson manifold. However, It is not the case that every Poisson bracket arises in this way. We shall see one such example below.

1.3 Examples

Example 8 (the plane). Consider the plane $\mathbb{R}^2$ with the usual coordinates $x, y : \mathbb{R}^2 \to \mathbb{R}$. The *standard symplectic structure on* $\mathbb{R}^2$ is $\omega = \mathrm{d}x \wedge \mathrm{d}y$.

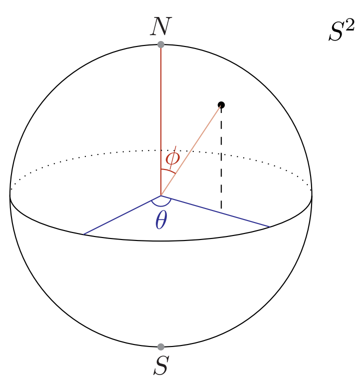
More generally, we can fix $N \geq 1$ and consider $\mathbb{R}^N$ with coordinates $(x_1, \ldots, x_N, y_1, \ldots, y_N)$. In terms of these coordinates, the standard symplectic structure on $\mathbb{R}^N$ is

$$\omega = \sum_{i=1}^{N} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Perhaps the most prominent example in these notes is the sphere $(S^2, \mathrm{d}\theta \wedge \mathrm{d}h)$, which will illustrate, for example:

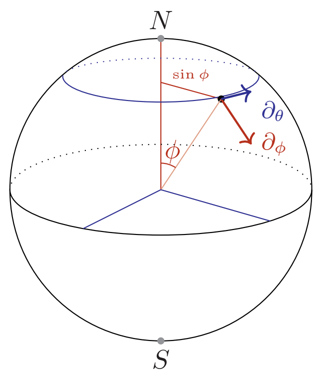
1. coadjoint orbits,
2. integral systems,
3. toric symplectic manifolds,
4. Kähler manifolds,
5. symplectic reduced spaces, in two distinct ways.

Example 9 (the sphere). Let S^2 be the unit sphere in $\mathbb{R}^3$, with the usual coordinates $\phi, \theta : S^2 \to \mathbb{R}$, given as illustrated below.



Observe the following:

1. The coordinate vector fields $\partial_\theta, \partial_\phi \in \mathfrak{X}\big(S^2\backslash\{N, S\}\big)$ are pointwise orthogonal.
2. Since ϕ measures the distance along S^2 to the north pole, ∂_ϕ has unit length at every point.
3. The integral curves of ∂_θ are circles of length $\sin\phi$, and so the vector field $\frac{1}{\sin\phi}\,\partial_\theta$ has unit length at every point.



It follows that $\big\{\partial_\phi, \frac{1}{\sin\phi}\,\partial_\theta\big\}$ is an orthonormal basis on $S^2\backslash\{N, S\}$ with metric dual basis $\{\mathrm{d}\phi, \sin\phi\,\mathrm{d}\theta\}$. In particular, the 2-form

$$\omega = \mathrm{d}\phi \wedge (\sin\phi\,\mathrm{d}\theta) = \sin\phi\,\mathrm{d}\phi \wedge \mathrm{d}\theta$$

is the usual volume form on S^2, and it is easy to see that ω is closed and nondegenerate. We will call ω the *standard symplectic structure on* S^2.

Now let $h : S^2 \to \mathbb{R}$ be the height function on S^2. For example, $h(N) = 1$ and $h(S) = -1$. Since $h = \cos\phi$, we have

$$\mathrm{d}h = -\sin\phi\,\mathrm{d}\phi,$$

and thus

$$\omega = -\mathrm{d}h \wedge \mathrm{d}\theta = \mathrm{d}\theta \wedge \mathrm{d}h.$$

Let us smoothly extend the coordinate vector field $\partial_\theta \in \mathfrak{X}\big(S^2\backslash\{N, S\}\big)$ to all of S^2 by defining $(\partial_\theta)_N = 0 \in T_N S^2$ and $(\partial_\theta)_S = 0 \in T_S S^2$. Since

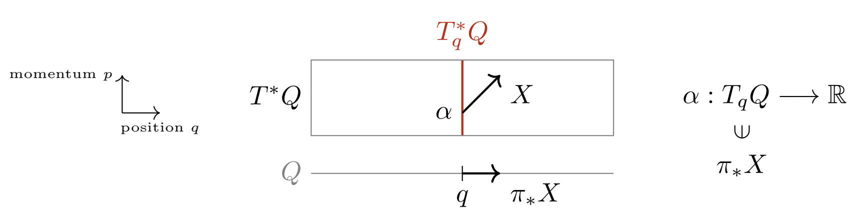
$$\iota_{\partial_\theta}\omega = \iota_{\partial_\theta}(\mathrm{d}\theta \wedge \mathrm{d}h) = \mathrm{d}h,$$

on $S^2\backslash\{N, S\}$, and since both sides vanish at N and S, we conclude that ∂_θ is the Hamiltonian vector field associated to h.

Example 10 (phase space). Let Q be any smooth manifold. We will show that the cotangent bundle T^*Q carries a natural symplectic structure $-\mathrm{d}\theta \in \Omega^2(T^*Q)$, where $\theta \in \Omega^1(Q)$ is given as follows. Fix $q \in Q$ and $\alpha \in T_q^*Q$. The value of $\theta_\alpha : T_\alpha(T^*Q) \to \mathbb{R}$ is defined by

$$\theta_\alpha(X) = \alpha(\pi_* X)$$

for every $X \in T_\alpha(T^*Q)$. Traditionally, the total space T^*Q is considered as the *momentum phase space* associated to the *configuration manifold* Q.



Definition 11. When a symplectic structure $\omega \in \Omega^2(M)$ is exact, so that $\omega = -\mathrm{d}\theta$ for some $\theta \in \Omega^1(M)$, we say that θ is a *symplectic potential* for ω, or a *Liouville* 1-*form* for ω.

In the next chapter, we will see that symplectic potentials never exist on closed manifolds.

We now introduce one of the most interesting examples.

Example 12 (Poisson structure on $\mathfrak{g}^*$). Let G be a semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. While not naturally a Lie algebra itself, the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is a key example of a Poisson manifold.

Here is the idea. For each point $\lambda \in \mathfrak{g}^*$, there is a canonical isomorphism of vector spaces

$$T_\lambda \mathfrak{g}^* \cong \mathfrak{g}^*,$$

and taking the dual of either side yields a canonical isomorphism

$$T^*_\lambda \mathfrak{g}^* \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}.$$

For $f, h \in C^\infty(\mathfrak{g}^*)$, define the value of $\{f, h\} \in C^\infty(\mathfrak{g}^*)$ at $\lambda \in \mathfrak{g}^*$ to be

$$\{f,h\}(\lambda) = \left\langle \lambda, \left[\tfrac{\delta f}{\delta \lambda}, \tfrac{\delta h}{\delta \lambda}\right]\right\rangle$$

where $\frac{\delta f}{\delta \lambda} \in \mathfrak{g}$ is determined, in terms of the above identifications, to be

$$\begin{array}{ccccc} T^*_\lambda \mathfrak{g}^* & \cong & (\mathfrak{g}^*)^* & \cong & \mathfrak{g} \\ \Cup & & & & \Cup \\ (\mathrm{d}f)_\lambda & & & & \frac{\delta f}{\delta \lambda} \end{array}$$

and similarly for $\frac{\delta h}{\delta \lambda} \in \mathfrak{g}$. The fact that $\{\,,\}$ is a Lie bracket on $\mathfrak{g}^*$ follows from the fact that $[\,,]$ is a Lie bracket on $\mathfrak{g}$. Moreover, $\{\,,\}$ is a bi-derivation as it factors through the exterior derivative $\mathrm{d} : C^\infty(\mathfrak{g}^*) \to \Omega^1(\mathfrak{g}^*)$. The reader may readily supply the technical details, if desired.

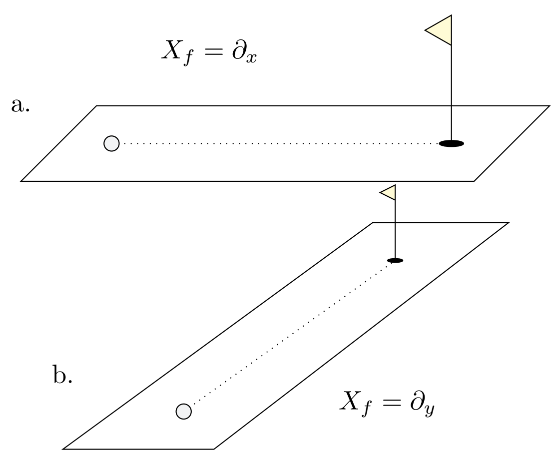
When we consider the coadjoint orbits $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$, later on, we will see that this Poisson bracket does not arise from a symplectic structure on $\mathfrak{g}^*$.

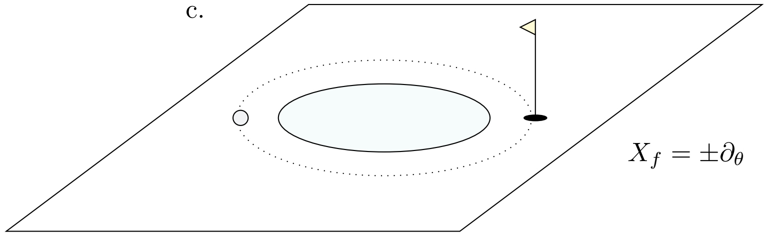
Exercises

1. Let (M, ω) be a symplectic manifold and $f \in C^\infty(M)$ a smooth function. Show that f is constant along the flow of $X_f \in \mathfrak{X}(M)$, that is, $X_f f = 0$.

   Hint. Show that $X_f f = \omega(X_f, X_f)$.

2. *Symplectic golf.* Consider the plane $\mathbb{R}^2$ equipped with the standard symplectic structure $\omega = \mathrm{d}x \wedge \mathrm{d}y$. In each illustration, find and sketch a Hamiltonian function $f \in C^\infty(\mathbb{R}^2)$ so that the Hamiltonian vector field $X_f \in \mathfrak{X}(\mathbb{R}^2)$ carries the ball into the hole with the flag.





Hint. For part c., consider a quadratic polynomial in x and y, and use the fact that $\partial_\theta = x\partial_y - y\partial_x$.

3. Consider the mapping

$$\begin{aligned} X : C^\infty(M) &\to \mathfrak{X}(M) \\ f &\mapsto X_f \end{aligned}$$

which sends a smooth function f to its associated Hamiltonian vector field X_f.

a. Show that the Poisson bracket $\{\,,\}$ is antisymmetric. That is, for all $f, h \in C^\infty(M)$, we have $\{f, h\} = -\{h, f\}$.

b. Show that $\{\,,\}$ is bilinear.

Hint. Show that $\{\,,\}$ is linear in the second argument, and use part a. to deduce that it is linear in the first argument.

c. Prove that $X : C^\infty(M) \to \mathfrak{X}(M)$ satisfies

$$X_{\{h,k\}} = [X_h, X_k]$$

for all $h, k \in C^\infty(M)$.

Hint. It requires to show that $\mathrm{d}\{h, k\} = \iota_{[X_h,X_k]}\omega$. You may find the identity $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$ helpful in this regard.

d. Prove that $\{\,,\}$ satisfies the Jacobi identity. That is, for all $f, h, k \in C^\infty(M)$,

$$\{f, \{h, k\}\} = \{\{f, h\}, k\} + \{h, \{f, k\}\}.$$

Hint. Part c. implies $X_{\{h,k\}}f = [X_h, X_k]f$.

e. Conclude that $\{\,,\}$ is a Lie bracket on $C^\infty(M)$, and that $X : C^\infty(M) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.

4. Show that the Poisson bracket $\{\,,\}$ is a *bi-derivation* on $C^\infty(M)$. That is,

$$\{f, hk\} = \{f, h\}k + h\{f, k\}$$

and

$$\{fh, k\} = \{f, k\}h + f\{h, k\}$$

for all $f, h, k \in C^\infty(M)$.

5. Suppose that (M, ω) is a compact symplectic manifold and that $f \in C^\infty(M)$ is a smooth function. Show that X_f has at least two vanishing points.

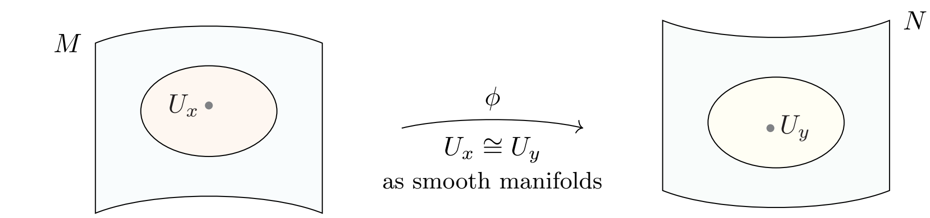
Hint. Show that X_f vanishes at the critical points of f.

Chapter 2

The Structure of Symplectic Manifolds

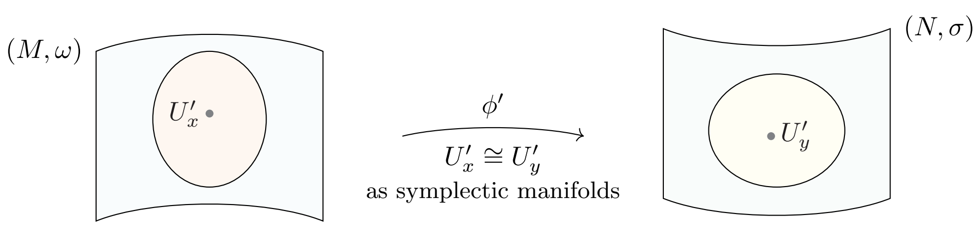
Having gained some intuition, we now begin a more technical development.

Consider two smooth manifolds M and N, each of dimension $k \geq 0$, and fix $x \in M$ and $y \in N$. In this setting, we can always find two neighborhoods $U_x \subseteq M$ and $U_y \subseteq N$, which are identical in the sense that there exists a diffeomorphism $\phi : U_x \to U_y$.



In other words, if you are given a very small neighborhood U of a smooth manifold M, then all you can know about M is its dimension. This is not the case for Riemannian manifolds: Indeed, if you are given any neighborhood $(U, g|_U)$ of a round sphere (S^2, g_{round}), then you know, for example, that it did not come from the flat plane $(\mathbb{R}^2, g_{\text{flat}})$.

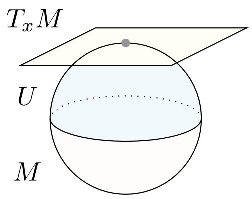
Now suppose that $\omega \in \Omega^2(M)$ and $\sigma \in \Omega^2(N)$ are symplectic structures on M and N. It turns out that we can always find neighborhoods $U'_x \subseteq M$ of x and $U'_y \subseteq M$ of y which are identified by a symplectomorphism $\phi' : (U_x, \omega|_{U'_x}) \to (U_y, \sigma|_{U'_y})$.



This is remarkable. A symplectic structure encodes no local information at all. Up close, the sphere $(S^2, \mathrm{d}\theta \wedge \mathrm{d}h)$ and the plane $(\mathbb{R}^2, \mathrm{d}x \wedge \mathrm{d}y)$ cannot be distinguished. This is the content of *Darboux's theorem*, which is the main result of this chapter.

In spite of this flexibility, it is not true that every manifold M possesses a symplectic structure $\omega \in \Omega^2(M)$. Locally, the nondegeneracy of ω requires that the dimension of M be even. Globally, while every contractible neighborhood $U \subseteq M$ on an even-dimensional manifold M admits a symplectic structure $\omega_U \in \Omega^2(U)$, topological considerations can prohibit the existence of a symplectic structure ω on M.

Our approach in this chapter is, by turns, infinitesimal, local, and global:



infinitesimal	*regarding*	tangent fibers T_xM
local		neighborhoods $U \subseteq M$
global		manifolds M

Key Points:

1. Symplectic manifolds and symplectic vector spaces are even-dimensional.
2. Any two symplectic vector spaces of the same dimension are symplectomorphic.
3. Any two symplectic manifolds of the same dimension are *locally* symplectomorphic.
4. A symplectic manifold (M^n, ω) possesses a canonical orientation and measure $\frac{\omega^n}{n!} \in \Omega^{2n}(M)$.
5. If M is compact, then $[\omega]$ defines a nontrivial cohomology class in $H^2(M;\mathbb{R})$.

2.1 Infinitesimal Structure

Consider a symplectic manifold (M, ω) and fix a point $x \in M$. Our aim in this section is to understand the tangent fiber T_xM as a vector space equipped with a nondegenerate alternating bilinear form $\omega_x : T_xM \times T_xM \to \mathbb{R}$.

To this end, we will temporarily disregard the manifold M, and fully turn our attention to the intrinsic structure of (T_xM, ω_x), that is, to the structure of a *symplectic vector space.* The most important properties are the following:

i. a symplectic vector space is necessarily even-dimensional, and

ii. any two symplectic vector spaces of the same dimension are isomorphic.

We also introduce some terminology that will be of use in the local case.

Definition 13. A *linear symplectic structure* on a vector space V is a nondegenerate alternating bilinear form $\omega : V \times V \to \mathbb{R}$. The pair (V, ω) is called a *symplectic vector space.*

Let us briefly review this terminology:

$$\begin{aligned} \textit{nondegeneracy:} \quad & \iota_u \omega \neq 0, \\ \textit{alternation:} \quad & \omega(u,v) = -\omega(v,u), \text{ and} \\ \textit{bilinearity:} \quad & \begin{cases} \omega(su+u',v) & = s\omega(u,v) + \omega(u',v), \\ \omega(u,sv+v') & = s\omega(u,v) + \omega(u,v'). \end{cases} \end{aligned}$$

where u, u', v and v' range over all points of V.

Example 14. Fix a vector space U. Let $V = U \oplus U^*$ and define the linear symplectic structure ω on V by

$$\omega(u + \phi, u' + \phi') = \phi'(u) - \phi(u')$$

for $u, u' \in U$ and $\phi, \phi' \in U^*$. From the perspective of classical mechanics, this is the canonical example of a symplectic vector space.

The terminology of maps is given in the natural way, as follows.

Definition 15. Let (V, ω) and (V', ω') be symplectic vector spaces. A linear map $\phi : V \to V'$ is called a *linear symplectic map* if $\omega = \phi^*\omega'$. If ϕ is additionally a linear isomorphism, then we say that ϕ is a *linear symplectomorphism* between (V, ω) and (V', ω').

We now introduce a very useful construction.

Definition 16. Let (V, ω) be a symplectic vector space. The *symplectic orthogonal* of the subspace $A \subseteq V$ the subspace

$$A^\omega = \{v \in V \mid \forall a \in A : \omega(a, v) = 0\}.$$

In other words, A^ω is the largest subspace of V which vanishes when paired with A under ω. The symplectic orthogonal is an essentially linear construction, in the sense that it finds no counterpart in the setting of symplectic manifolds. It will prove to be an important tool in the proof of the symplectic reduction theorem, which we will see later on.

Definition 17 (Distinguished subspaces)**.** Let (V, ω) be a symplectic vector space. A subspace $A \subseteq V$ is said to be

- *symplectic*, if ω defines a linear symplectic structure on A,
- *isotropic*, if ω vanishes on A,
- *coisotropic*, if ω vanishes on A^ω,
- *Lagrangian*, if $A = A^\omega$.

More concisely:

symplectic	$A \cap A^\omega = 0$
isotropic	$A \subseteq A^\omega$
coisotropic	$A^\omega \subseteq A$
Lagrangian	$A = A^\omega$

This characterization provides the following important structure result for symplectic vector spaces.

Lemma 18 (Existence of symplectic bases)**.** *Let (V, ω) be a symplectic vector space has a basis of the form $\{u_1, \dots, u_n, v_1, \dots, v_n\} \subseteq V$ for some $n \geq 1$, where*

$$\omega(u_i, v_i) = 1, \qquad \omega(u_i, u_j) = \omega(v_i, v_j) = 0,$$

for all $i, j \leq n$.

To prove this result, we proceed constructively. Specifically, we will obtain a disjoint collection of symplectic subspaces $V_i = \langle u_i, v_i \rangle \subseteq V$ such that $V = V_1 \oplus \cdots \oplus V_k$. To achieve this, we will define a descending chain of subspaces $\{V_{(k)}\}_{k=1}^n$ designed to satisfy $V_{(k)} = V_k \oplus V_{k+1} \cdots \oplus V_n$.

Proof. Let $u_1 \in V$ be nonzero. The nondegeneracy of ω implies that there is a $\tilde{v}_1 \in V$ such that $\omega(u_1, \tilde{v}_1) \neq 0$. Put $v_1 = \tilde{v}_1 / \omega(u_1, \tilde{v}_1) \in V$ and observe that $\omega(u_1, v_1) = 1$. Since the subspace $V_1 = \langle u_1, v_1 \rangle$ is symplectic, the symplectic orthogonal $V_{(2)} = V_1^\omega$ is also symplectic, and $V = V_1 \oplus V_{(2)}$. If $V_{(2)} = 0$ then we are done. Otherwise, we repeat this procedure and obtain $V_{(2)} = V_2 \oplus V_{(3)}$ for a suitable $V_2 = \langle u_2, v_2 \rangle \subseteq V_{(2)}$. As V is finite-dimensional, this procedure must terminate with $V_{(n+1)} = 0$ for some $n \geq 0$. □

A basis $\mathcal{B} = \{u_1, \dots, u_k, v_1, \dots, v_k\} \subseteq V$ of the above form is called a *symplectic basis* of (V, ω). Observe that

$$\omega = u_1^* \wedge v_1^* + \cdots + u_n^* \wedge v_n^*,$$

where $\mathcal{B}^* = \{u_1^*, \dots, u_k^*, v_1^*, \dots, v_k^*\} \subseteq V^*$ is the dual basis to $\mathcal{B}$. While we have shown that symplectic bases always exist, the degree of free choice in the construction of our proof correctly suggests that they are never unique.

Proposition 19. *If (V, ω) and (V', ω') are symplectic vector spaces, then*

i. V is even-dimensional,

ii. (V, ω) and (V', ω') are linearly symplectomorphic if and only if $\dim V = \dim V'$.

Proof. Let $\mathcal{B} = \{u_1, \ldots, u_m, v_1, \ldots, v_m\} \subseteq V$ and $\mathcal{B}' = \{u'_1, \ldots, u'_n, v'_1, \ldots, v'_n\} \subseteq V'$ be symplectic bases, as provided by Lemma 18.

i. $\dim V = |\mathcal{B}| = 2n$.

ii. If (V, ω) and (V', ω') are linearly symplectomorphic, then they are isomorphic as vector spaces. In particular, $\dim V = \dim V'$.

Now suppose $\dim V = \dim V'$. There is a unique linear isomorphism $\phi : V \to V'$ sending $u_i \mapsto u'_i$ and $v_i \mapsto v'_i$ for all $i \leq m$. Since ω agrees with $\phi^*\omega'$ on the members of $\mathcal{B}$, since $\mathcal{B}$ is a basis, and since ω is bilinear, it follows that $\omega = \phi^*\omega'$. Thus, (V, ω) and (V', ω') are linearly symplectomorphic.

□

In the next section, we will show that symplectic manifolds are equipped with canonically defined orientations and volume forms. Given a symplectic basis $\mathcal{B} = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$, consider the linear volume element

$$u_1^* \wedge v_1^* \wedge \cdots \wedge u_n^* \wedge v_n^* \ \in \Lambda^{2n} V^*.$$

In the following lemma, we establish that this element depends only on (V, ω). In particular, it does not depend on the choice of symplectic basis $\mathcal{B}$.

Lemma 20. *If (V, ω) is a symplectic vector space, and if $\mathcal{B} \subseteq \{u_1, \ldots, u_n, v_1, \ldots, v_n\} \subseteq V$ is any symplectic basis, then*

$$\frac{\omega^{2n}}{n!} = u_1^* \wedge v_1^* \wedge \cdots \wedge u_n^* \wedge v_n^* \ \in \Lambda^{2n} V^*,$$

where $\mathcal{B}^* = \{u_1^*, \ldots, u_n^*, v_1^*, \ldots, v_n^*\} \subseteq V^*$ is the dual basis to $\mathcal{B}$.

Proof. Put $\alpha_i = u_i^* \wedge v_i^*$ for each $i \leq n$. Thus,

$$\omega = \alpha_1 + \cdots + \alpha_n.$$

Since $\alpha_i^2 = 0$ and $\alpha_i \alpha_j = \alpha_j \alpha_i$, we accumulate like terms to obtain

$$\omega^2 = \sum_{i<j} 2\, \alpha_i \alpha_j$$

and

$$\omega^3 = \sum_{i<j<k} 6\, \alpha_i \alpha_j \alpha_k.$$

Indeed, since each term $\alpha_{i_1} \cdots \alpha_{i_k}$ is invariant under all $|S_k| = k!$ permutations of $\{i_1, \ldots, i_k\}$, we obtain the general formula

$$\omega^k = \sum_{i_1 < \cdots < i_k} k!\, \alpha_{i_1} \cdots \alpha_{i_k}.$$

In particular, $\omega^n = n!\, \alpha_1 \cdots \alpha_n$. □

We have shown that $\frac{\omega^n}{n!}$ is a linear volume element on V, which we will call the *canonical linear symplectic volume element* on (V, ω), and is characterized by the property that $\frac{\omega^{2n}}{n!}(u_1, v_1, \ldots, u_n, v_n) = 1$ for every symplectic basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n\} \subseteq V$. In terms of the exponential map, we have

$$e^\omega = 1 + \frac{\omega}{1!} + \frac{\omega^2}{2!} + \ldots + \frac{\omega^n}{n!} \ \in \Lambda^* V^*,$$

so that the canonical linear symplectic volume element is given by

$$(e^\omega)_{[2n]} = \frac{\omega^n}{n!} \ \in \Lambda^{2n} V^*.$$

2.2 Local Structure

We turn now from the tangent fiber T_xM to a sufficiently small neighborhood U of $x \in M$. Our analysis will be motivated and informed by the linear symplectic structure of (T_xM, ω_x).

First let us extend the classification scheme of Definition 17 to the smooth setting.

Definition 21 (Distinguished submanifolds)**.** Let (M, ω) be a symplectic manifold. A submanifold $N \subseteq M$ is called a *symplectic (resp. isotropic, coisotropic, Lagrangian) submanifold* if T_xN is a symplectic (resp. isotropic, coisotropic, Lagrangian) subspace of (T_xM, ω_x) for every $x \in N$.

The next result follows easily from our study of the linear situation.

Proposition 22. *Every symplectic manifold (M, ω) is even-dimensional.*

Proof. Since (T_xM, ω_x) is a symplectic vector space, Proposition 19 part i. asserts that $\dim M = \dim T_xM$ is even. ☐

The most profound result of the local setting is arguably *Darboux's theorem*. Informally, this states that any two symplectic manifolds of the same dimension are locally indistinguishable.

Lemma 23 (Existence of symplectic coordinates)**.** *Let (M^{2n}, ω) be a symplectic manifold and fix $x \in M$. There is a neighborhood $U \subseteq M$ of x and a coordinate chart $(x_1, \ldots, x_n, y_1, \ldots, y_n) : U \to \mathbb{R}$ such that*

$$\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$$

on U.

We proceed constructively, by analogy with Lemma 18. The function y_1 is chosen arbitrarily, and the function x_1 is defined so that the coordinate vector field ∂_{x_i} is the Hamiltonian function X_{y_1}. The argument is repeated to obtain the remaining coordinates $x_2, y_2, \ldots, x_n, y_n$.

Proof sketch. Take a small neighborhood $U_1 \subseteq M$ of x and choose a function $y_1 \in C^\infty(U_1)$ so that $\mathrm{d}y_1$ is nonvanishing. Let $x_1 \in C^\infty(U_1)$ be any solution to the ordinary differential equation $X_{y_1} x_1 = 1$. If M is a surface, then $\partial_{x_1} = X_{y_1}$ and $\partial_{y_1} = -X_{x_1}$, so that $\omega(\partial_{x_1}, \partial_{y_1}) = \omega(X_{y_i}, -X_{x_i}) = 1$ at every point of $U = U_1$ and we are done.

Otherwise, let $y_2 \in C^\infty(U_2)$ be chosen on a neighborhood $U_2 \subseteq U_1$ of x so that $\mathrm{d}y_2$ is nonvanishing and $\partial_{x_1} y_2 = \partial_{y_1} y_2 = 0$, and choose any solution $x_2 \in C^\infty(U_2)$ of the system of equations $\partial_{x_1} x_2 = \partial_{y_1} x_2 = 0$ and $\partial_{y_2} x_2 = 1$. Continuing in this manner yields the system of coordinates $x_1, y_1, \ldots, x_n, y_n \in C^\infty(U_n)$ on a neighborhood $U = U_n$ of x. ☐

A coordinate chart of the above form $(x_1, \ldots, x_n, y_1, \ldots, y_n) : U \to \mathbb{R}$ is called a *symplectic coordinate chart.*

Theorem 24 (Darboux)**.** *If (M, ω) and (N, σ) are symplectic manifolds of the same dimension, then (M, ω) and (N, σ) are locally symplectomorphic.*

Proof. Lemma 23 implies that (M, ω) and (N, σ) are each locally symplectomorphic to $(\mathbb{R}^{2n}, \sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i)$. Therefore, they are locally symplectomorphic to each other. ☐

In fact, Lemma and Theorem 24 are both known as Darboux's theorem.

2.3 Global Structure

In this final section, we consider the global structure of a symplectic manifold (M, ω). Since the Darboux theorem implies that (M, ω) has no local invariants, the study of symplectic manifolds is sometimes known as *symplectic topology*.

Proposition 25. *The form $\omega^n \in \Omega^{2n}(M)$ defines a volume element on M.*

Proof. Lemma 20 implies that $\omega_x^n \in \Lambda^{2n} T_x^* M$ is nonzero at every $x \in M$. That is, the top-degree form $\omega^n \in \Omega^{2n}(M)$ is nowhere vanishing and, consequently, defines a volume form on M. □

Fix $x \in M$. The space of highest-degree elements $\Lambda^{2n} T_x^* M$ is a 1-dimensional vector space. By removing the zero element, we obtain a space $\Lambda^{2n} T_x^* M \backslash \{0\}$ with two connected components. These two connected components comprise the fiber of the *orientation bundle* $\widetilde{M} \to M$ at x. That is,

$$\widetilde{M}_x = \left(\Lambda^{2n} T_x^* M \backslash \{0\}\right) / \sim$$

where $\eta \sim \eta'$ if and only if η and $\eta' \in \Lambda^{2n} T^* M \backslash \{0\}$ lie in the same connected component or, equivalently, if $\eta = \lambda \eta'$ for some positive real number $\lambda > 0$. An *orientation* $[M] \in H_{2n}(M, \mathbb{Z}_2)$ of M is a section of the orientation bundle $\widetilde{M} \to M$.

Corollary 26. *A symplectic manifold (M, ω) possesses a canonical orientation and measure.*

Proof. Since $\omega^n \in \Omega^{2n}(M)$ is nowhere vanishing, it descends to a section $[\omega]_\sim$ of the orientation bundle $\widetilde{M} \to M$. This endows M with the measure $U \mapsto \int_U \frac{\omega^n}{n!}$, where we resolve the sign ambiguity according to the condition that $\int_M \phi \, \frac{\omega^n}{n!} > 0$ for every compactly supported smooth function $\phi \in C^\infty(M)$. □

We will always assume that (M, ω) is equipped with the canonical orientation.

Corollary 27. *If M is compact, then $[\omega]^k \in H^{2k}(M; \mathbb{R})$ is nonzero for all $k \leq n$.*

Proof. Suppose for a contradiction that $\omega^k = \mathrm{d}\alpha$ for some $\alpha \in \Omega^{2k-1}(M)$. An application of Stokes' theorem yields

$$\int_M \omega^n = \int_M \mathrm{d}(\alpha \wedge \omega^{n-k}) = 0.$$

This provides the desired contradiction, since Proposition 25 asserts that $\frac{\omega^n}{n!}$ is a volume form on M. □

Exercises

1. Let (V, ω) be a symplectic vector space and let $U \subseteq V$ be any subset. Confirm that $U^\omega = \{v \in V \mid \forall u \in U : \omega(u, v) = 0\}$ is a subspace of V.

2. Use the original definitions of symplectic, isotropic, coisotropic, and Lagrangian subspaces A of (V, ω) to verify the following alternative characterizations.

symplectic	$A \cap A^\omega = 0$
isotropic	$A \subseteq A^\omega$
coisotropic	$A^\omega \subseteq A$
Lagrangian	$A = A^\omega$

3. Let A be any subspace of (V, ω). Show that

   i. $\dim A + \dim A^\omega = \dim V$,

   ii. $A = A^{\omega\omega}$.

Hint. For part i., consider the linear map $v \mapsto \iota_v \omega$ from V to V^*, and use the rank–nullity theorem. For part ii., show that $A \subseteq A^{\omega\omega}$, and deduce from part i. that $\dim A = \dim A^{\omega\omega}$.

4. Prove that the Klein bottle does not admit a symplectic structure.

5. Show that S^2 is the only sphere S^k, $k \geq 2$, which admits a symplectic structure.

   Hint. Recall that the real cohomology of S^k is given by

$$H^i(S^k, \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = 0, k \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 3

Group Actions and Hamiltonian Manifolds

We have seen that the smooth functions f on M induce infinitesimal symmetries X_f of (M, ω). We also know, more generally, that the symmetries of a smooth mathematical structure are encoded in the action of a Lie group G. Our goal in this chapter is to combine these two ideas. In particular, we will show how the action of G may be described infinitesimally in terms of the assignment of Hamiltonian vector fields on (M, ω).

Since a family of Hamiltonian vector fields describes *local* symmetries, while the action of a Lie group encodes *global* symmetries, our first task is to resolve this difference in scope. We achieve this by means of the *fundamental vector fields* associated to a Lie group action. The *moment map* then naturally arises as a bridge between the symmetries encoded in the the action of G on M, and those described by the assignment of Hamiltonian vector fields $f \mapsto X_f$.

The construction of the moment map, which completes the definition of a *Hamiltonian manifold* (M, ω, G, μ), is so fundamental to our investigation that the remainder of this chapter is devoted to examples. The first broad class of examples that we consider comprise the classical phase spaces T^*Q associated to configuration manifolds Q. The second consists of the coadjoint orbits $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ which, as we shall see, carry both a natural symplectic structure and a natural action of G.

Before we begin, let us make a general remark on group actions in the symplectic setting.

Definition 28. Fix a Lie group G and a symplectic manifold (M, ω). An action of G on M is said to be a *symplectic action* if G preserves ω. In this case, we say that G acts on (M, ω).

A symplectic action $G \curvearrowright (M, \omega)$ describes the symmetries of (M, ω) as a symplectic manifold. Symplectic actions thus form the natural class of actions associated to the symplectic category. Unless otherwise stated, all actions on symplectic manifolds in these notes are assumed to be symplectic.

Key Points:

1. A comoment map $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ for the action $G \curvearrowright (M, \omega)$ describes the assignment of fundamental vector fields $\xi \mapsto \underline{\xi}$ in terms of the assignment of Hamiltonian dynamics on (M, ω).

2. A moment map for $G \curvearrowright (M, \omega)$ is a smooth function $\mu : M \to \mathfrak{g}^*$ such that (i) $\mathrm{d}\mu_\xi = \iota_{\underline{\xi}}\omega$ and (ii) $\xi \mapsto \mu_\xi$ is a homomorphism of Lie algebras.

3. If Q is a smooth manifold, then an action $G \curvearrowright Q$ yields a Hamiltonian manifold $(T^*Q, -\mathrm{d}\theta, G, \mu)$.

4. The coadjoint orbit $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ naturally possesses the structure of a Hamiltonian manifold $(\mathcal{O}_\lambda, \omega, G, \mu)$, where $\mu : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$ is the inclusion.

3.1 Fundamental Vector Fields

One of the most interesting and useful features of the smooth setting is that our global constructions often possess infinitesimal counterparts or descriptions. Here are some examples:

global	*infinitesimal*
manifold	tangent bundle
diffeomorphism	vector field
Lie group	Lie algebra

Consider the action of a Lie group G on a manifold M. Formally, this is a homomorphism from G to the group of diffeomorphisms Diff M, subject to the familiar the smoothness conditions. The aim of this section is to describe the infinitesimal counterpart of the action $G \curvearrowright M$, namely, the *assignment of fundamental vector fields* $\xi \mapsto \underline{\xi}$.

Definition 29. Fix an element $\xi \in \mathfrak{g}$. The *fundamental vector field* $\underline{\xi} \in \mathfrak{X}(M)$ is given by

$$\underline{\xi}_x = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-t\xi} \cdot x \in T_x M$$

at each $x \in M$.

The negative sign in the above expression is a common source of confusion. The next result offers some justification.

Proposition 30. *The assignment of fundamental vector fields*

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{X}(M) \\ \xi &\mapsto \underline{\xi} \end{aligned}$$

is a homomorphism of Lie algebras.

Proof. Fix $x \in M$. Observe that

$$(\underline{\xi}\,\underline{\eta} f)(x) = \underline{\xi}_x \underbrace{\left(y \mapsto -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(e^{t\eta} \cdot y) \right)}_{\in C^\infty(M)} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\substack{s=0\\t=0}} f(e^{t\eta} e^{s\xi} \cdot x)$$

implies

$$[\underline{\xi}, \underline{\eta}]_x = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\substack{s=0\\t=0}} e^{t\eta} e^{s\xi} \cdot x \;\; - \;\; \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\substack{s=0\\t=0}} e^{t\xi} e^{s\eta} \cdot x,$$

while

$$[\xi, \eta] = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\substack{s=0\\t=0}} e^{s\xi} e^{t\eta} e^{-s\xi}$$

yields

$$\begin{aligned} \underline{[\xi, \eta]}_x &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\substack{s=0\\t=0}} e^{s\xi} e^{t\eta} e^{-s\xi} \cdot x \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\substack{s=0\\t=0}} e^{s\xi} e^{t\eta} \cdot x \; + \; \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\substack{s=0\\t=0}} e^{t\eta} e^{s\xi} \cdot x. \end{aligned}$$

In particular, $[\underline{\xi}, \underline{\eta}] = \underline{[\xi, \eta]}$. □

One sometimes sees the fundamental vector field induced by $\xi \in \mathfrak{g}$ define by $\underline{\xi}'_x = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} e^{t\xi} \cdot x$. The assignment $\xi \mapsto \underline{\xi}'$ is an *anti*-Lie algebra homomorphism. That is, $\underline{[\xi,\eta]}' = -[\underline{\xi}', \underline{\eta}']$ for all $\xi, \eta \in \mathfrak{g}$. Informally, we might identify the source of the negative sign to be in the inversion of the order of the terms ξ and η in the identity

$$(\underline{\xi}\,\underline{\eta} f)(x) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\substack{s=0\\t=0}} f(e^{t\eta}e^{s\xi} \cdot x),$$

as shown in the proof of Proposition 30.

Proposition 30 asserts that, with our conventions, the assignment of fundamental vector fields is a Lie algebra action of $\mathfrak{g}$ on M. Let us recall what this means.

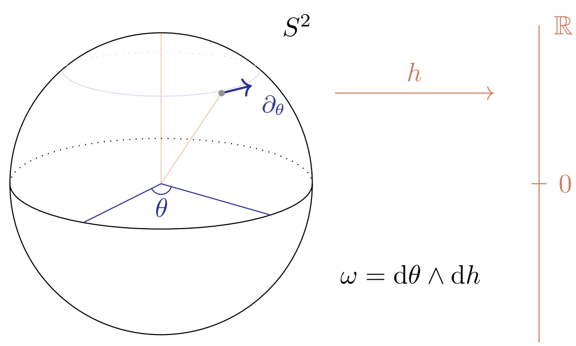
Definition 31. Fix a Lie algebra $\mathfrak{g}$. A *Lie algebra action* of $\mathfrak{g}$ on M is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{X}(M)$.

The fundamental vector fields $\underline{\xi} \in \mathfrak{X}(M)$ completely describe the action of the identity component of G. Specifically, the transformation of M induced by $e^{\xi} \in G$ is the unit-time flow of the vector field $-\underline{\xi} \in \mathfrak{X}(M)$.

In these notes, we will adopt a common notational convention. Given $\xi \in \mathfrak{g}$ and $\alpha \in \Omega^*(M)$, we will omit the underline and write $\iota_\xi \alpha$ for $\iota_{\underline{\xi}}\alpha \in \Omega^*(M)$ and $\mathcal{L}_\xi \alpha$ for $\mathcal{L}_{\underline{\xi}}\alpha \in \Omega^*(M)$.

3.2 Introducing the Moment Map

Consider the sphere.



We have seen that the height function $h : S^2 \to \mathbb{R}$ is a Hamiltonian function for the vector field $\partial_\theta \in \mathfrak{X}(S^2)$. The vector field ∂_θ, in turn, generates a one-parameter group of transformations of (S^2, ω) in the form of rotations about the vertical axis. Informally, we might say that the function h generates the rotations of the sphere.

Let us try to capture this intuition in terms of group actions and fundamental vector fields. Let the circle group $\mathrm{U}(1) = e^{\mathrm{i}\mathbb{R}} \subseteq \mathbb{C}$ act on S^2 by rotations about the vertical axis, so that $e^{\mathrm{i}t} \in S^1$ induces a rotation by $2\pi t$ radians in the negative θ direction. The vector field induced by $\mathrm{i} \in \mathrm{i}\mathbb{R} \cong \mathfrak{u}(1)$ is ∂_θ. More generally, the vector field induced by $\xi = \mathrm{i}t \in \mathrm{i}\mathbb{R} \cong \mathfrak{u}(1)$ is given by $\underline{\xi} = t\partial_\theta$. In particular, the induced vector fields of $\mathrm{i}t \in \mathfrak{u}(1)$ is the Hamiltonian vector field for the function $t \cdot h \in C^\infty(M)$.

Thus, to say that h generates the action of $\mathrm{U}(1)$ is to say that the infinitesimal symmetry $X_h \in \mathfrak{X}(M)$ $h \in C^\infty(M)$ coincides with the infinitesimal action of $\mathrm{i} \in \mathfrak{u}(1)$. Furthermore, as the action of a Lie group G on a manifold M is locally described by the fundamental vector fields $\xi \mapsto \underline{\xi}$, it is sometimes the case that each fundamental vector field $\underline{\xi}$ may be identified as the Hamiltonian vector field associated to some function $\tilde{\mu}(\xi) \in C^\infty(M)$. This identification takes the form of a *comoment map*.

Definition 32. A *comoment map* for the action of G on (M, ω) is a homomorphism of Lie algebras $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ such that $\tilde{\mu}(\xi)$ is a Hamiltonian function for the fundamental vector field $\underline{\xi} \in \mathfrak{X}(M)$.

That is, $\tilde{\mu}$ is a lift of the assignment of fundamental vector fields to the space of smooth functions, in the sense that we have a commutative diagram,

$$
\begin{array}{ccccc}
 & & C^\infty(M) & & f \\
 & \overset{\tilde{\mu}}{\nearrow} & \downarrow & & \downarrow \\
\mathfrak{g} & \longrightarrow & \mathfrak{X}(M) & & X_f \\
\xi & \longmapsto & \underline{\xi} & &
\end{array}
$$

in the category of Lie algebras and Lie algebra homomorphisms. In this way, a comoment map factors the induced action of a Lie algebra through the Hamiltonian dynamics of (M, ω).

It turns out that the comoment map $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ is not very convenient to work with. We prefer to repackage $\tilde{\mu}$ in the form of a *moment map* $\mu : M \to \mathfrak{g}^*$, given as follows.

First, some notation: Denote by $\langle\, , \rangle$ the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and write $\lambda_\xi = \langle \lambda, \xi \rangle$ for $\lambda \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$. Given $\xi \in \mathfrak{g}$ and $\alpha \in \Omega^*(M)$, recall that the expression $\iota_\xi \alpha$ is shorthand for $\iota_{\underline{\xi}} \alpha \in \Omega^*(M)$.

Definition 33. A *moment map* for the action of G on (M, ω) is a smooth function $\mu : M \to \mathfrak{g}^*$ such that

i. $\mathrm{d}\mu_\xi = \iota_\xi \omega$ for every $\xi \in \mathfrak{g}$,

ii. the assignment $\xi \mapsto \mu_\xi$ is a homomorphism of Lie algebras.

In this case, we say that $G \curvearrowright (M, \omega)$ is a *Hamiltonian action*, and that (M, ω, G, μ) is a *Hamiltonian manifold*.

That is, $\mu : M \to \mathfrak{g}^*$ is a moment map precisely when $\xi \mapsto \mu_\xi$ is a comoment map for $G \curvearrowright (M, \omega)$. The moment map $\mu : M \to \mathfrak{g}^*$ encodes a family of smooth functions $(\mu_\xi)_{\xi \in \mathfrak{g}}$ which collectively describe a Lie group action $G \curvearrowright (M, \omega)$ by identifying the fundamental vector fields $\underline{\xi}$ with the Hamiltonian vector fields $X_{\mu_\xi} \in \mathfrak{X}(M)$.

$f : M \to \mathbb{R}$	*generates a*	one-parameter transformation, along the flow of $-X_f$
$\mu : M \to \mathfrak{g}^*$		Lie group action, according to the identity $\underline{\xi} = X_{\mu_\xi}$

Every moment map $\mu : M \to \mathfrak{g}^*$ for $G \curvearrowright (M, \omega)$ defines a comoment map $\xi \mapsto \mu_\xi$. The converse is also true: If $\tilde{\mu}$ is a comoment map for $G \curvearrowright (M, \omega)$, then $\mu(x)(\xi) = \tilde{\mu}(\xi)(x)$ defines a moment map μ for $G \curvearrowright (M, \omega)$. Note that the expression $\mu(x)(\xi)$ is equal to $\langle \mu(x), \xi \rangle$. It may be helpful to clarify the domain of each component:

$$\underbrace{\mu(x)}_{\mathfrak{g}^*} \quad \underbrace{(\xi)}_{\mathfrak{g}} \quad = \quad \underbrace{\tilde{\mu}(\xi)}_{C^\infty(M)} \quad \underbrace{(x)}_{M}$$

Indeed, $\mu : M \to C^\infty(M)$ and $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ are just two different perspectives on the two-point function

$$
\begin{aligned}
\mathfrak{g} \times M &\longrightarrow \mathbb{R} \\
(\xi, x) &\longmapsto \mu(x)(\xi) = \tilde{\mu}(\xi)(x).
\end{aligned}
$$

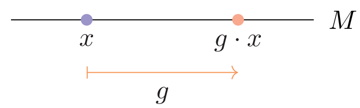
In light of this equivalence, we consider the moment map $\mu : M \to \mathfrak{g}^*$ both as a smooth function on M, and in terms of its associated comoment map $\xi \mapsto \mu_\xi$.

3.3 Examples from Classical Mechanics

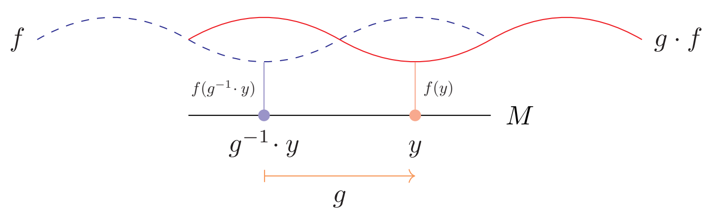
Let Q be a smooth manifold equipped with the action of a Lie group G. The action of G on Q induces an action of G on T^*Q according to the rule

$$(g \cdot \alpha_q)(X_{g \cdot q}) = \alpha(g^{-1} \cdot X_{gq}),$$

where $g \in G$, $q \in Q$, $\alpha \in T_q^*Q$, and $X \in T_qQ$. Let us first make sense of this formula, particularly the appearance of the inverse element g^{-1}. Suppose G acts on a smooth manifold M.



The induced action on $C^\infty(M)$ is given by $(g \cdot f)(y) = f(g^{-1} \cdot y)$ for $f \in C^\infty(M)$ and $g \in G$.



Informally, we push f forward by pulling M back. The transition from $G \curvearrowright Q$ to $G \curvearrowright T^*Q$ is similar.

Recall that the canonical symplectic structure on T^*Q is given by $\omega = -\mathrm{d}\theta$, where $\theta \in \Omega^1(T^*Q)$ is the canonical 1-form.

Proposition 34. *The assignment $\mu : T^*Q \to \mathfrak{g}^*$ given by*

$$\mu_\xi(\alpha_q) = -\alpha_q(\underline{\xi}_q),$$

for $q \in Q$, $\alpha_q \in T_qQ$ and $\xi \in \mathfrak{g}$, is a moment map for the induced action of G on T^*Q.

Proof. We will show that

i. $\mathrm{d}\mu_\xi = \iota_\xi \omega$ for all $\xi \in \mathfrak{g}$,
ii. $\xi \mapsto \mu_\xi$ is a Lie algebra homomorphism.

Since $\pi_* \underline{\xi}_\alpha = -\underline{\xi}_q$ for $\alpha \in T_q^*Q$ and $\xi \in \mathfrak{g}$, we have $\mu_\xi(\alpha) = \theta_\alpha(\underline{\xi}_\alpha)$. That is, $\mu_\xi = \iota_\xi \theta$.

i. Since θ is preserved by the lift of any smooth transformation of Q, it follows that $\mathcal{L}_\xi \theta = 0$, and consequently that $\mathrm{d}\mu_\xi = \mathrm{d}\iota_\xi\theta = -\iota_\xi \mathrm{d}\theta = \iota_\xi \omega$.

ii. Proposition 30 implies that $\iota_{\underline{[\xi,\eta]}} = \iota_{[\underline{\xi},\underline{\eta}]} = [\mathcal{L}_{\underline{\xi}}, \iota_{\underline{\eta}}]$, so that

$$\mu_{[\xi,\eta]} = \iota_{[\xi,\eta]}\theta = [\mathcal{L}_\xi, \iota_\eta]\,\theta = \mathcal{L}_\xi \iota_\eta \theta = \iota_\xi \mathrm{d}\iota_\eta \theta.$$

The result follows as

$$\{\mu_\xi, \mu_\eta\} = -\omega(\underline{\xi}, \underline{\eta}) = \mathrm{d}\theta(\underline{\xi}, \underline{\eta}) = -\iota_\xi \iota_\eta \mathrm{d}\theta.$$

□

3.4 Coadjoint Orbits

Fix a Lie group G.

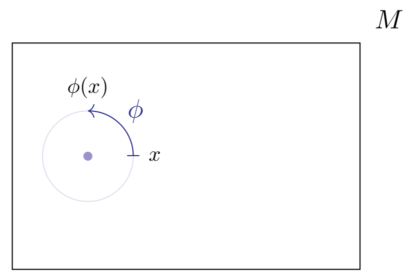
Definition 35. The action of *conjugation* of G on G is given by $g \cdot h = ghg^{-1}$ for $g, h \in G$.

The action of conjugation is the natural action of G on G *as a group of transformations*. Let us explain this comment.

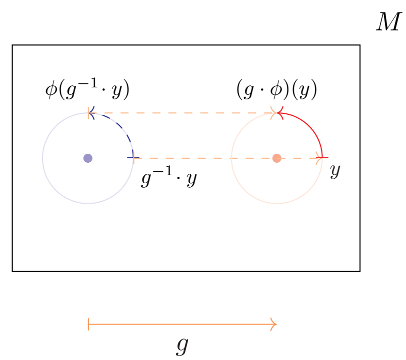
If G acts on a smooth manifold M, then the natural action of G on the space of diffeomorphisms $\operatorname{Diff} M$ is given by

$$(g \cdot \phi)(x) = g \cdot \phi(g^{-1} \cdot x),$$

where $\phi \in \operatorname{Diff} M$, $g \in G$, and $x \in M$. To see why this is, consider the case in which M is a plane and $\phi \in \operatorname{Diff} M$ is a rotation.



If $g \in G$ induces a translation on M, then the formula for $g \cdot \phi$ appears to "translate" the rotation ϕ.



We now return to conjugation. Left multiplication associates each element $h \in G$ with a transformation $\phi_h \in \operatorname{Diff} G$ of $M = G$, given by $\phi_h(x) = hx$ for $h, x \in G$. Let us write $\bar{G}$ for the collection of transformations $(\phi_g)_{g \in G}$. Clearly, G and $\bar{G}$ are naturally isomorphic as groups. Now, left multiplication also describes an action G on $M = G$, given by $g \cdot x = gx$ for $g, x \in G$. Applying the above ideas, we obtain

$$(g \cdot \phi_h)(x) = g \cdot \phi_h(g^{-1} \cdot x) = ghg^{-1}x = \phi_{ghg^{-1}}(x).$$

Therefore, the action of left multiplication of G on G induces the action of conjugation of G on $\bar{G}$. In fact, a similar argument shows that the action $\phi : G \to \operatorname{Sym} X$ of any group G on any set X induces the action of conjugation of G on $\bar{G} = \phi(G)$.

An infinitesimal counterpart to the action of conjugation $G \curvearrowright G$ is the *adjoint action* $G \curvearrowright \mathfrak{g}$.

Definition 36. Let $\phi : G \to \operatorname{Aut} G$ denote the action of conjugation. The *adjoint action* of G on $\mathfrak{g}$ is given by

$$\operatorname{Ad}_g \xi = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi_g(e^{t\xi})$$

for $g \in G$ and $\xi \in \mathfrak{g}$.

The adjoint action of G on $\mathfrak{g}$ induces an action of G on the dual space $\mathfrak{g}^* \subseteq C^\infty(\mathfrak{g})$.

Definition 37. The *coadjoint action* of G on $\mathfrak{g}^*$ is given by

$$\langle \operatorname{Ad}_g^* \lambda, \xi \rangle = \langle \lambda, \operatorname{Ad}_{g^{-1}} \xi \rangle$$

for $g \in G$, $\lambda \in \mathfrak{g}^*$, and $\xi \in \mathfrak{g}$.

That is, $(g \cdot \lambda)(\xi) = \lambda(g^{-1} \cdot \xi)$, in accordance with our discussion above.
We now arrive at the main construction of this section.

Definition 38. The *coadjoint orbit* through $\lambda \in \mathfrak{g}^*$ is the orbit $\mathcal{O}_\lambda = G \cdot \lambda \subseteq \mathfrak{g}^*$ of the coadjoint action of G on $\mathfrak{g}^*$.

Since the action of G is smooth, the orbit $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ is a smooth submanifold.

Proposition 39. *Fix $\lambda \in \mathfrak{g}^*$. We have*

$$\mathcal{O}_\lambda \cong G/\operatorname{Stab_G} \lambda$$

as smooth manifolds.

Proof sketch. The principal map

$$\begin{aligned} G &\to \mathcal{O}_\lambda \\ g &\mapsto g \cdot \lambda \end{aligned}$$

descends to a smooth bijection

$$G/\operatorname{Stab_G} \lambda \xrightarrow{\sim} \mathcal{O}_\lambda.$$

□

This describes the smooth structure of $\mathcal{O}_\lambda$. Before we define the natural symplectic structure on $\mathcal{O}_\lambda$, we first review some terminology.

Recall the adjoint action of G on $\mathfrak{g}$. This action admits a further infinitesimal expression, taking the form of the identically-named *adjoint action* $\mathfrak{g} \curvearrowright \mathfrak{g}$.

Definition 40. The *adjoint action* of $\mathfrak{g}$ on $\mathfrak{g}$ is given by

$$\operatorname{ad}_\eta \xi = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \operatorname{Ad}_{e^{s\eta}} \xi$$

for $\eta, \xi \in \mathfrak{g}$.

A key property of the adjoint ad is that

$$\operatorname{ad}_\xi \eta = [\xi, \eta]$$

for all $\xi, \eta \in \mathfrak{g}$. The coadjoint action of G on $\mathfrak{g}^*$ is define similarly.

Definition 41. The *coadjoint action* of $\mathfrak{g}$ on $\mathfrak{g}^*$ is given by

$$\operatorname{ad}_\xi^* \lambda = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \operatorname{Ad}_{e^{s\eta}}^* \xi$$

Thus,

$$\langle \operatorname{ad}_\eta^* \lambda, \xi \rangle = \langle \lambda, \operatorname{ad}_{-\eta} \xi \rangle = -\langle \lambda, [\eta, \xi] \rangle$$

for $\lambda \in \mathfrak{g}^*$ and $\eta, \xi \in \mathfrak{g}$. All together, we have introduced the following actions:

$G \curvearrowright G$	conjugation	$\text{—} : G \to \operatorname{Aut} G$
$G \curvearrowright \mathfrak{g}$	adjoint action	$\operatorname{Ad} : G \to \operatorname{GL} \mathfrak{g}$
$\mathfrak{g} \curvearrowright \mathfrak{g}$	adjoint action	$\operatorname{ad} : G \to \mathfrak{gl}\, \mathfrak{g}$
$G \curvearrowright \mathfrak{g}^*$	coadjoint action	$\operatorname{Ad}^* : G \to \operatorname{GL} \mathfrak{g}$
$\mathfrak{g} \curvearrowright \mathfrak{g}^*$	coadjoint action	$\operatorname{ad}^* : G \to \mathfrak{gl}\, \mathfrak{g}$

When speaking, we sometimes call Ad the "big" adjoint action and ad the "little" adjoint action, and similarly for Ad^* and ad^*.

We are now ready to define the natural symplectic structure on $\mathcal{O}_\lambda$.

Definition 42. Fix $\lambda \in \mathfrak{g}^*$. Define the 2-form $\omega \in \Omega^2(\mathcal{O}_\lambda)$ by

$$\omega(\underline{\xi}_\tau, \underline{\eta}_\tau) = -\langle \tau, [\xi, \eta] \rangle,$$

for all $\tau \in \mathcal{O}_\lambda$ and $\xi, \eta \in \mathfrak{g}$.

Proposition 43. *The form $\omega \in \Omega^2(\mathcal{O}_\lambda)$ is a symplectic structure on $\mathcal{O}_\lambda$.*

Proof. We will show that ω is

i. well-defined,
ii. closed,
iii. nondegenerate.

Fix a point $\tau \in \mathcal{O}_\lambda$.

i. Since $\mathcal{O}_\lambda$ is an orbit of G, every tangent vector $X \in T_\tau \mathcal{O}_\lambda$ is of the form $\underline{\xi}_\tau$ for some $\xi \in \mathfrak{g}$. We must show that if $\xi, \xi' \in G$ satisfy $\underline{\xi}_\tau = \underline{\xi'}_\tau$, then $\langle \tau, [\xi, \eta] \rangle = \langle \tau, [\xi', \eta] \rangle$ for all $\eta \in \mathfrak{g}$. This follows as

$$\operatorname{ad}^*_{(\xi - \xi')} \tau = \underline{\xi - \xi'}_\tau = 0$$

implies that

$$\big\langle \tau, [\xi - \xi', \eta] \big\rangle = -\langle \operatorname{ad}^*_{(\xi - \xi')} \tau, \eta \rangle = 0.$$

ii. Using Proposition 30, we derive that

$$\begin{aligned} d\omega(\underline{\xi}, \underline{\eta}, \underline{\zeta}) &= \underline{\xi}\, \omega(\underline{\eta}, \underline{\zeta}) - \underline{\eta}\, \omega(\underline{\xi}, \underline{\zeta}) + \underline{\zeta}\, \omega(\underline{\xi}, \underline{\eta}) \\ &\qquad - \omega([\underline{\xi}, \underline{\eta}], \underline{\zeta}) + \omega([\underline{\xi}, \underline{\zeta}], \underline{\eta}) - \omega([\underline{\eta}, \underline{\zeta}], \underline{\xi}) \\ &= \langle \operatorname{ad}^*_\xi \tau, [\eta, \zeta] \rangle - \langle \operatorname{ad}^*_\eta \tau, [\xi, \zeta] \rangle + \langle \operatorname{ad}^*_\zeta \tau, [\xi, \eta] \rangle \\ &\qquad \langle \tau, [[\xi, \eta], \zeta] \rangle - \langle \tau, [[\xi, \zeta], \eta] \rangle + \langle \tau, [[\eta, \zeta], \xi] \rangle \\ &= -\langle \tau, [\xi, [\eta, \zeta]] \rangle + \langle \tau, [\eta, [\xi, \zeta]] \rangle - \langle \tau, [\zeta, [\xi, \eta]] \rangle \\ &\qquad -\langle \tau, [[\xi, \eta], \zeta] \rangle + \langle \tau, [[\xi, \zeta], \eta] \rangle - \langle \tau, [[\eta, \zeta], \xi] \rangle \\ &= 0, \end{aligned}$$

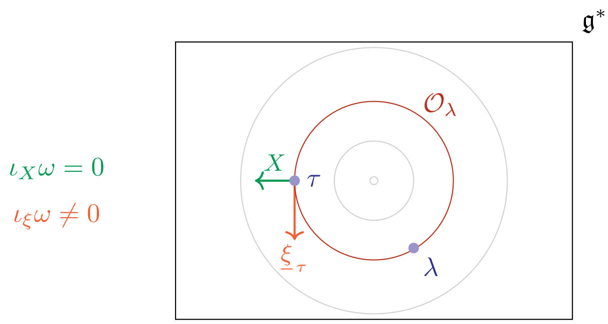
where we have treated $\tau \in \mathcal{O}_\lambda$ as an indeterminate.

iii. Let $\xi \in \mathfrak{g}$ and suppose that $\omega(\underline{\xi}_\tau, \underline{\eta}_\tau) = 0$ for all $\eta \in \mathfrak{g}$. It follows that

$$\langle \operatorname{ad}^*_\xi \tau, \eta \rangle = -\langle \tau, [\xi, \eta] \rangle = 0$$

for all $\eta \in \mathfrak{g}$, so that $\underline{\xi}_\tau = \operatorname{ad}^*_\xi \tau = 0$.

$\square$



The natural action of G on a coadjoint orbit $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ is, of course, the restriction of the coadjoint action Ad^*. As we now show, this action is Hamiltonian and admits a particularly simple moment map.

Proposition 44. *Fix $\lambda \in \mathfrak{g}^*$. The inclusion $\mu : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$ is a moment map for the natural action of G on $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$.*

Proof. We will show that

i. $\mathrm{d}\mu_\xi = \iota_\xi\omega$ for all $\xi \in \mathfrak{g}$,
ii. $\mu : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$ is G-equivariant.

i. We have
$$\mathrm{d}\mu_\xi(\underline{\eta}) = \langle \mathrm{d}\mu(\underline{\eta}), \xi\rangle = -\langle \mu, [\eta, \xi]\rangle = \omega(\underline{\eta}, \underline{\xi}).$$

ii. This follows by the construction of the action on $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$.

□

Exercises

1. Let (M, ω, G, μ) be a Hamiltonian system. Show that
$$\langle \mu_* X, \xi\rangle = -\omega(X, \underline{\xi}_x)$$
for all $X \in T_xX$, and $\xi \in \mathfrak{g}$.

   Hint. Evaluate both sides of the equality $\mathrm{d}\mu_\xi = \iota_\xi\omega$ on X.

2. We have seen that action of $U(1)$ by rotations on the sphere $(S^2, \mathrm{d}\theta \wedge \mathrm{d}h)$ is Hamiltonian. Consider the action of $U(1)$ on the torus $T^2 = (\mathbb{R}^2/\mathbb{Z}^2, \mathrm{d}x \wedge \mathrm{d}y)$ given by
$$e^{2\pi\, t} \cdot \big[(x, y)\big] = \big[(t + x, y)\big]$$
for all $x, y, t \in \mathbb{R}$.

   i. Show that the action $U(1) \curvearrowright T^2$ is symplectic.

   ii. Show that this action is *not* Hamiltonian. That is, show that it does not admit any moment map.

3. According to our definition, a Hamiltonian action is assumed to be symplectic. Give an example of an action of a Lie group G on a symplectic manifold (M, ω) which satisfies

   i. $\mathrm{d}\mu_\xi = \iota_\xi\omega$,

   ii. $\xi \mapsto \mu_\xi$ is a Lie algebra homomorphism but is *not* symplectic.

Hint. Consider the two-point group $G = \mathbb{Z}_2$.

4. Consider a symplectic action of G on (M, ω), and suppose that the smooth function $\nu : M \to \mathfrak{g}^*$ satisfies

   i. $\mathrm{d}\nu_\xi = \iota_\xi \omega$, for all $\xi \in \mathfrak{g}$,

   ii. $\mathrm{Ad}_g^* \, \nu(x) = \nu(g \cdot x)$ for all $g \in G$ and $x \in M$, that is, ν is G-equivariant.

   Show that ν is a moment map for the action of G on (M, ω).

   Hint. It remains to show that the assignment $\xi \mapsto \nu_\xi = \langle \nu, \xi \rangle$ is a morphism of Lie algebras. In particular, that

   $$\nu_{[\xi,\eta]} = \{\nu_\xi, \nu_\eta\}$$

   for all $\xi, \eta \in \mathfrak{g}$. Compare each side of this equation with the identity for G-equivariance in condition ii. above. Recall that $\{\nu_\xi, \nu_\eta\} = \mathcal{L}_{\underline{\xi}} \, \nu_\eta$ and $[\xi, \eta] = \mathrm{ad}_\xi \, \eta$.

5. Let (M, ω) be a symplectic manifold and suppose that $\omega = -\mathrm{d}\theta$ for some $\theta \in \Omega^1(M)$. Suppose, furthermore, that the Lie group G acts on M and preserves θ.

   i. Show that the action of G on (M, ω) is symplectic.

   ii. Find a moment map $\mu : M \to \mathfrak{g}^*$ for the action of G.

   Hint. You may wish to consult Proposition 2 in the notes.

6. Let G be an abelian Lie group and fix an element $\lambda \in \mathfrak{g}^*$. Show that the coadjoint orbit $\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$ is equal to the singleton $\{\lambda\}$.

Chapter 4

Reduction

The purpose of this chapter is to prove the symplectic reduction theorem.

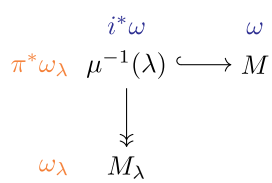
Consider a manifold M, a Lie group action of G on M, and a smooth function $f : M \to N$. There are two standard ways to remove degrees of freedom of M. First, we may take the quotient M/G. Second, we may take the preimage $f^{-1}(y)$ of a regular value $y \in N$. A Hamiltonian manifold (M, ω, G, μ) suggests both of these procedures: a quotient by the action of G, and a preimage of a regular value $\lambda \in \mathfrak{g}^*$ under the moment map $\mu : M \to \mathfrak{g}^*$.

When $\dim G \geq 1$, it turns out that ω never descends to a symplectic structure on M/G, and never restricts to a symplectic structure on $\mu^{-1}(\lambda) \subseteq M$. The resulting 2-form is always degenerate. Intriguingly, ω does a symplectic structure when both these two operations are used together. Denote by $G_\lambda \subseteq G$ the stabilizer subgroup of $\lambda \in \mathfrak{g}^*$ under the coadjoint action of G on $\mathfrak{g}^*$. Subject to mild conditions on the action $G \curvearrowright M$ and the moment map $\mu : M \to \mathfrak{g}^*$, the *reduced space* $M_\lambda = \mu^{-1}(\lambda)/G_\lambda$ inherits a natural symplectic structure from (M, ω). This is the content of the *Marsden–Weinstein–Meyer symplectic reduction theorem*, due to Marsden–Weinstein [10] and [11]. Here we present the statement due to Marsden and Weinstein.

Theorem 45 (Symplectic reduction)**.** *Let (M, ω, G, μ) be a Hamiltonian manifold with G-equivariant moment map μ, and let $\lambda \in \mathfrak{g}^*$ be a regular value of μ. If G_λ acts freely and properly on $\mu^{-1}(\lambda) \subseteq M$, then there is a unique symplectic structure ω_λ on the* reduced space *$M_\lambda = \mu^{-1}(\lambda)/G_\lambda$ such that*

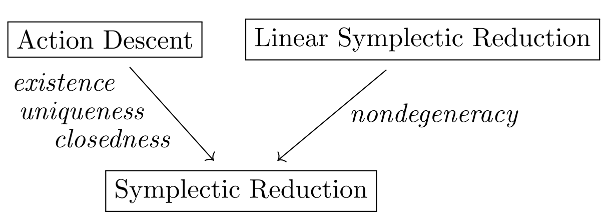
$$\pi^*\omega_\lambda = i^*\omega$$

where $i : \mu^{-1}(\lambda) \to M$ is the inclusion and $\pi : \mu^{-1}(\lambda) \to M_\lambda$ is the projection.



Our proof utilizes two lemmas:

1. the *action descent lemma* ensures that $i^*\omega$ descends to a unique, closed ω_λ on M_λ,
2. the *linear symplectic reduction lemma* guarantees that ω_λ is nondegenerate.

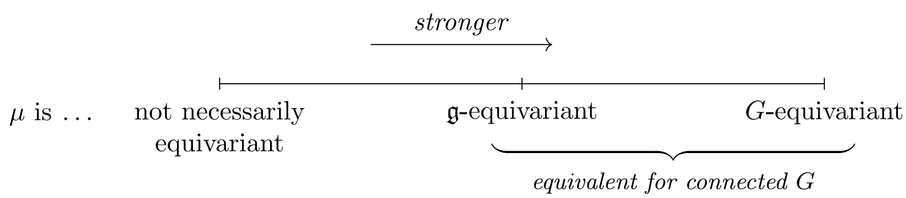


From these lemmas, the symplectic reduction theorem follows easily.

Key Points:

1. Symplectic reduction is a process for taking a Hamiltonian manifold (M, ω, G, μ) and an admissible value $\lambda \in \mathfrak{g}^*$, and obtaining a new symplectic manifold $(M_\lambda, \omega_\lambda)$ where $M_\lambda = \mu^{-1}(\lambda)/G_\lambda$.
2. Informally, symplectic reduction removes conjugate degrees of freedom (x_i, y_i) together, so that the resulting space remains symplectic.
3. The complex projective spaces $(\mathbb{C}P^n, \omega_{FS})$ is a symplectic reduction of $(\mathbb{C}^{n+1}, \sum_i \mathrm{d}x \wedge \mathrm{d}y)$ under the action of scalar multiplication by $U(1) \subseteq \mathbb{C}^*$.
4. The symplectic reduction of the cotangent bundle T^*Q, equipped with an action lifted from $G \curvearrowright Q$, is the cotangent bundle $T^*(Q/G)$.

Remark. A review of the literature turns up varying definitions of the moment map. In every case, $\mu_\xi \in C^\infty(M)$ is required to be a Hamiltonian function for $\underline{\xi} \in \mathfrak{X}(M)$ for every $\xi \in \mathfrak{g}$. In some sources, this is the defining condition. Others require μ to be G-equivariant. Our condition, that $\xi \mapsto \mu_\xi$ is a homomorphism of Lie algebras, or equivalently that μ is $\mathfrak{g}$-equivariant, is intermediate between these two.



The reader is also advised that the names "action descent lemma" and "linear symplectic reduction lemma" are not in standard usage.

4.1 The Idea of Reduction

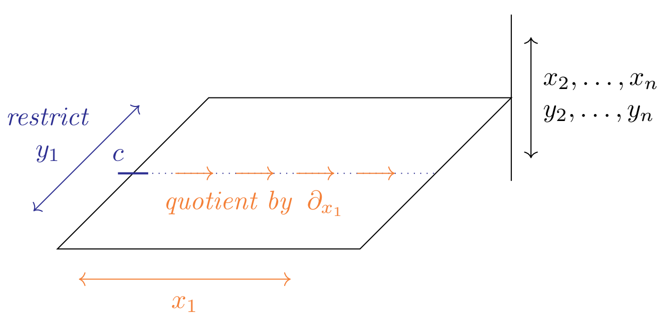
Before we begin the proof, let us first discuss the underlying ideas.

Consider a symplectic coordinate chart $(x_1, \ldots, x_n, y_1, \ldots y_n)$ on a neighborhood U of (M, ω). Suppose we wish to remove the degrees of freedom on U corresponding to the coordinates x_1 and y_1.

$$\omega = \underbrace{\mathrm{d}x_1 \wedge \mathrm{d}y_1}_{\text{to be removed}} + \mathrm{d}x_2 \wedge \mathrm{d}y_2 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$$

Our approach is to

i. *restrict* the coordinate $y_1 : U \to \mathbb{R}$ to the fixed value $c \in \mathbb{R}$,
ii. *quotient* by the flow of ∂_{x_1}.



We are left with the remaining degrees of freedom $x_2, \ldots, x_n, y_2, \ldots, y_n$ and the symplectic structure

$$\omega_c = \mathrm{d}x_2 \wedge \mathrm{d}y_2 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

It is important that conjugate pairs (x_1, y_1) are removed together. If we only restrict y_1 to $c \in \mathbb{R}$, then the induced 2-form on $y^{-1}(c) \subseteq U$ would be degenerate in the x_1 direction. If we only quotient by the flow of ∂_{y_1}, then the induced two form on $U/\langle \partial_{x_1} \rangle$ would be degenerate in the y_1 direction.

This informal discussion captures the essential idea of symplectic reduction:

Reduce (M, ω) by restricting and quotienting conjugate degrees of freedom.

To see how these local ideas extend to the global setting, replace the coordinate function y_1 with an arbitrary smooth function $f \in C^\infty(M)$, and replace the coordinate vector field ∂_{x_1} with the Hamiltonian vector field $X_f \in \mathfrak{X}(M)$. Recall from our proof of Darboux's theorem that, away from the critical points of f, we may always locally extend $y_1 = f$ to a system of symplectic coordinates $(x_i, y_i)_{i \leq k}$ with $\partial_{x_1} = X_f$.

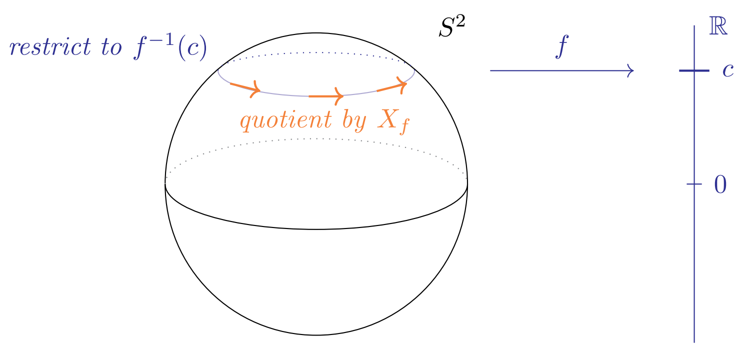
local	*global*
y_1	f
∂_{x_1}	X_f

Adapting the previous local approach, we proceed to

i. *restrict* to the preimage $f^{-1}(c) \subseteq M$ for a fixed regular value $c \in \mathbb{R}$,
ii. *quotient* by the flow of the Hamiltonian vector field $X_f \in \mathfrak{X}(M)$

That c is a regular value ensures that $f^{-1}(c) \subseteq M$ is a smooth submanifold. Note that X_f preserves $f^{-1}(c)$ since $X_f f = 0$. To ensure a smooth quotient, we could, for example, introduce the condition that X_f generates a free circle action on $f^{-1}(c)$.

We illustrate this on the sphere $(S^2, \mathrm{d}\theta \wedge \mathrm{d}h)$ with f equal to the height function h and $X_f = \partial_\theta$.

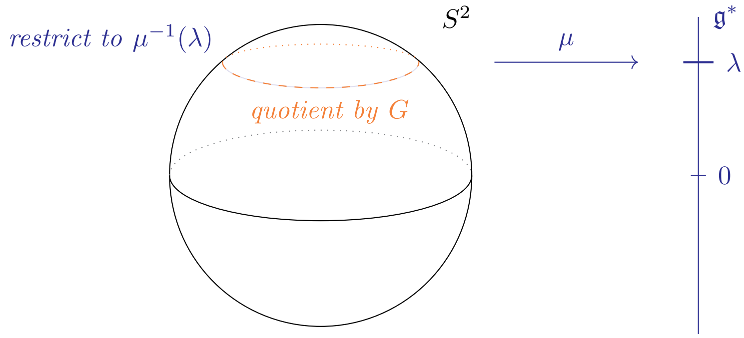


This procedure removes from M, first, those directions in which f varies; second, those directions tangent to X_f. In our illustration on S^2, we are left with a point $f^{-1}(c)/S^1$, on which $\omega \in \Omega^2(S^2)$ induces the trivial symplectic structure $\omega_c = 0$.

Fix a Hamiltonian manifold (M, ω, G, μ) and suppose that G is connected so that the fundamental vector fields $\underline{\xi} \in \mathfrak{X}(M)$ generate the action of G. Recall that the moment map $\mu : M \to \mathfrak{g}^*$ encodes a family of functions $\mu_\xi \in C^\infty(M)$, with associated Hamiltonian vector fields $X_{\mu_\xi} = \underline{\xi}$. Our idea now is to restrict to the preimage $\mu_\xi^{-1}(\lambda_\xi)$ of every function $\mu_\xi \in C^\infty(M)$ for $\xi \in \mathfrak{g}$, and take the quotient with respect to every Hamiltonian vector field $X_{\mu_\xi} = \underline{\xi}$. That is, we

i. *restrict* to the preimage $\mu^{-1}(\lambda) \subseteq M$ for a fixed regular value $\lambda \in \mathfrak{g}^*$,

ii. *quotient* by the action of G.



The symplectic reduction theorem provides broad conditions under which the restriction $\omega|_{\mu^{-1}(\lambda)}$ descends to a symplectic structure ω_λ on $M_\lambda = \mu^{-1}(\lambda)/G$.

4.2 Linear Symplectic Reduction

Turning to the setting of symplectic vector spaces, we establish a result which will model the infinitesimal situation of the general symplectic reduction theorem. Looking ahead, we have in mind the identifications

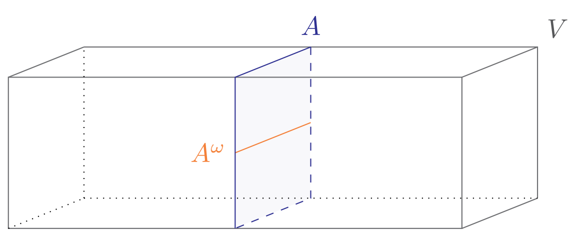
- $V = T_x M$, for some point $x \in \mu^{-1}(\lambda)$,
- $A = T_x\mu^{-1}(\lambda)$ the tangent space to the preimage $\mu^{-1}(\lambda) \subseteq M$,
- $A^\omega = \underline{\mathfrak{g}}_x$ the subspace spanned by the action of $\mathfrak{g}_\lambda$ at x.

The distribution $\underline{\mathfrak{g}} = \{\underline{\xi}_x \mid x \in M,\, \xi \in \mathfrak{g}\} \subseteq TM$ is called the *fundamental distribution* associated to the action $G \curvearrowright M$

Lemma 46 (Linear symplectic reduction)**.** *Let (V, ω) be a symplectic vector space. If $A \subseteq V$ is a subspace, then ω descends to a linear symplectic structure $\bar\omega$ on $\bar A = A/(A \cap A^\omega)$, where*

$$\bar\omega([a],[b]) = \omega(a,b)$$

for $a, b \in A$.



Proof. We will show that

i. $\bar\omega$ is well-defined,
ii. $\bar\omega$ is nondegenerate.

This will complete the proof, since $\bar\omega$ is clearly alternating and bilinear.

i. If $a', b' \in A$ with $[a] = [a']$ and $[b] = [b']$, then $a - a', b - b' \in A^\omega$. Thus,

$$\omega(a, b) = \omega\big(a' + \underbrace{(a - a')}_{A^\omega}, b' + \underbrace{(b - b')}_{A^\omega}\big) = \omega(a', b'),$$

and it follows that $\bar{\omega} : \bar{A} \times \bar{A} \to \mathbb{R}$ is well-defined.

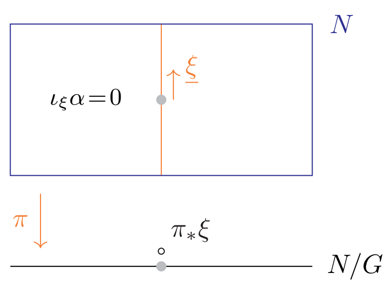
ii. Fix $a \in A$. If $\bar{\omega}([a], [b]) = \omega(a, b) = 0$ for all $b \in A$, then $a \in A^\omega$, so that $[a] = 0 \in \bar{A}$.

□

4.3 Descending Forms

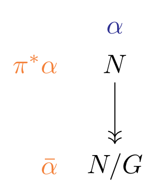
We now consider the general question of when a differential form $\alpha \in \Omega^*(N)$ descends under the quotient map $\pi : N \to N/G$ for a group action $G \curvearrowright N$ to a differential form on N/G. Applying this result to the action of the stabilizer subgroup $G_\lambda \subseteq G$ on the level set $N = \mu^{-1}(\lambda) \subseteq M$ will enable us to obtain everything in the conclusion of the symplectic reduction theorem, with the exception of the nondegeneracy of ω_λ.

Definition 47. Let $\pi : N \to P$ be a smooth map of manifolds. A tangent vector $X \in TN$ is said to be *π-vertical* if $\pi_* X = 0$. A form $\alpha \in \Omega^*(N)$ is called *π-horizontal* if $\iota_X \alpha = 0$ for all π-vertical $X \in TN$. In particular, if $\pi : N \to N/G$ is the quotient map for a smooth action $G \curvearrowright N$, then we say that α is *G-horizontal* if $\iota_\xi \alpha = 0$ for all $\xi \in \mathfrak{g}$.



Informally, $\alpha \in \Omega^*(M)$ is horizontal when it has no extension in the vertical directions.

Lemma 48 (Action descent)**.** *Suppose the Lie group G acts freely and properly on the smooth manifold N and let $\pi : N \to N/G$ be the quotient map. If $\alpha \in \Omega^*(N)$ is horizontal and invariant, then there is a unique form $\bar{\alpha} \in \Omega^*(N/G)$ such that $\pi^*\bar{\alpha} = \alpha$. Moreover, $\mathrm{d}\alpha = 0$ if and only if $\mathrm{d}\bar{\alpha} = 0$.*



Proof. We will assume without loss of generality that α is homogeneous. That is, $\alpha \in \Omega^k(N)$ for some $k \geq 0$. Since the action of G is free and proper, it follows that N/G is smooth, and that $\pi : N \to N/G$ is a surjective submersion. Define

$$\bar{\alpha}_{\pi x}(\pi_* X_1, \ldots, \pi_* X_k) = \alpha_x(X_1, \ldots, X_k),$$

where $x \in N$ and $X_1, \ldots, X_k \in T_x N$. Since $\pi : N \to N/G$ is surjective, this defines $\bar{\alpha}$ at every point $x \in N$. Clearly $\pi^*\bar{\alpha} = \alpha$, and it follows that $\mathrm{d}\alpha = 0$ if $\bar{\alpha} = 0$. It remains to prove that

i. $\bar{\alpha}$ is well-defined and smooth,
ii. $\bar{\alpha}$ is unique,
iii. $\mathrm{d}\alpha = 0$ implies $\mathrm{d}\bar{\alpha} = 0$.

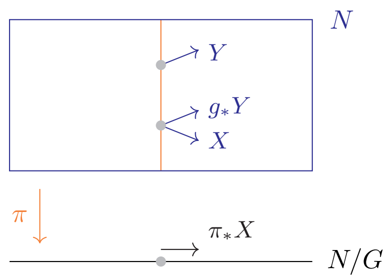
We prove each in turn.

i. Clearly, $\bar{\alpha}$ is well-defined when $\alpha \in \Omega^0(N)$ is a G-invariant smooth function on N. Now suppose that $k > 0$ and assume the claim to be true for forms of lower degree. Fix $x \in N$ and $X \in T_xN$. We will show that $\iota_{\pi_* X}\bar{\alpha} \in \Lambda^{k-1}T^*_{\pi x}(N/G)$ is well-defined.

   If $Y \in T_yN$ with $\pi_* X = \pi_* Y$, then there is a $g \in G$ with $gy = x$. From $\pi_* X = \pi_* g_* Y$ it follows that $X - g_* Y \in T_xN$ is vertical. Therefore,

$$\begin{aligned} \iota_X \alpha &= \iota_{g_* Y}\alpha, && \text{since } \alpha \text{ is horizontal,} \\ &= \iota_Y \alpha, && \text{since } \alpha \text{ is invariant.} \end{aligned}$$

   We conclude that $\iota_{\pi_* X}\bar{\alpha}$ identifies a well-defined element of $\Lambda^{k-1}T^*_{\pi x}(N/G)$.



   To see that α is smooth, observe that every vector field on N/G is of the form $\pi_* Z \in \mathfrak{X}(N/G)$ for some G-invariant vector field $Z \in \mathfrak{X}(N)$. For any choice of $Z_1, \dots, Z_k \in \mathfrak{X}(N)$, the function

$$\bar{\alpha}(\pi_* Z_1, \dots, \pi_* Z_k) = \pi_*\big[\alpha(Z_1, \dots, Z_k)\big]$$

   is smooth on N/G, since $\alpha(Z_1, \dots, Z_k)$ is a G-invariant smooth function on N.

ii. Fix $x \in N$. Since $\pi : N \to N/G$ is submersive, the derivative $\pi_* : T_xN \to T_{\pi x}(N/G)$ is surjective, and thus the dual map $\pi^* : T^*_{\pi x}(N/G) \to T^*_xN$ is injective. Consequently, $\bar{\alpha}_{\pi x}$ is the unique element of $T^*_{\pi x}(N/G)$ satisfying $\pi^*\bar{\alpha}_{\pi x} = \alpha_x$.

iii. Suppose $\mathrm{d}\alpha = 0$. Using again the fact that $\pi^* : T^*_{\pi x}(N/G) \to T^*_xN$ is injective, it follows from the identity $\pi^*\mathrm{d}\bar{\alpha} = \mathrm{d}\pi^*\bar{\alpha} = 0$ that $(\mathrm{d}\bar{\alpha})_{\pi x} = 0$.

□

We are now ready to prove the symplectic reduction theorem.

Proof of the reduction theorem. Since $\lambda \in \mathfrak{g}^*$ is a regular value of $\mu : M \to \mathfrak{g}^*$, the preimage $\mu^{-1}(\lambda) \subseteq M$ is a smooth submanifold. Since $\mu : M \to \mathfrak{g}^*$ is equivariant, we have

$$\mu(g \cdot x) = g \cdot \mu(x) = \lambda$$

for all $g \in G_\lambda$ and $x \in \mu^{-1}(\lambda)$. This implies that G_λ preserves $\mu^{-1}\lambda$. As ω is G-invariant, and as G_λ is a subgroup of G, it follows that $i^*\omega$ is G_λ-invariant. Moreover, observe that $i^*\omega$ is G_λ-horizontal since

$$\begin{aligned} \iota_\xi i^*\omega &= i^*\iota_\xi\omega, && \text{since } i : \underline{\mathfrak{g}} \hookrightarrow T\mu^{-1}(0) \text{ is an inclusion,} \\ &= i^*\mathrm{d}\mu_\xi, && \text{since } \mu_\xi \text{ is a Hamiltonian function for } \underline{\xi}, \\ &= \mathrm{d}i^*\mu_\xi \\ &= \mathrm{d}\langle\lambda, \xi\rangle, && \text{since } \mu \text{ takes the constant value } \lambda \text{ on } \mu^{-1}(\lambda), \\ &= 0 \end{aligned}$$

for every $\xi \in \mathfrak{g}_\lambda$. It follows from Lemma 48 that there is a unique, closed 2-form $\omega_\lambda \in \Omega^2(M_\lambda)$ satisfying $\pi^*\omega_\lambda = i^*\omega$.

It remains to show that ω_λ is nondegenerate. Fix $x \in \mu^{-1}(\lambda)$ and $X \in T_xM$. From the equality

$$\omega(\underline{\xi}, X) = \iota_{\xi}\omega(X) = \mathrm{d}\mu_\xi(X) = \langle \mu_* X, \xi\rangle, \qquad \xi \in \mathfrak{g}$$

we deduce that

$$\begin{aligned} X \in \underline{\mathfrak{g}}^\omega &\iff \langle \mu_* X, \mathfrak{g}\rangle = \omega(\underline{\mathfrak{g}}, X) = 0 \\ &\iff \mu_* X = 0 \\ &\iff X \in T_x\mu^{-1}(\lambda). \end{aligned}$$

That is, $\underline{\mathfrak{g}}^\omega = T_x\mu^{-1}(\lambda)$, from which it follows that

$$T_x\mu^{-1}(\lambda)^\omega = \underline{\mathfrak{g}}^{\omega\omega} = \underline{\mathfrak{g}}.$$

An application of Lemma 46 yields that ω_λ is nondegenerate as a bilinear form on

$$T_{\pi x}M_\lambda \cong T_{\pi x}\big(\mu^{-1}(\lambda)/G_\lambda\big) \cong T_x\mu^{-1}(\lambda)/(T_x\mu^{-1}(\lambda) \cap \underline{\mathfrak{g}}_x).$$

□

4.4 Examples

Example 49 ($\mathbb{C}P^n$). Define the *standard symplectic structure* on the real manifold $\mathbb{C}^{n+1}$ to be

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_{n+1} \wedge \mathrm{d}y_{n+1},$$

where x_i and y_i are obtained from the standard complex coordinates

$$(x_1 + \mathrm{i}y_1, \ldots, x_{n+1} + \mathrm{i}y_{n+1}).$$

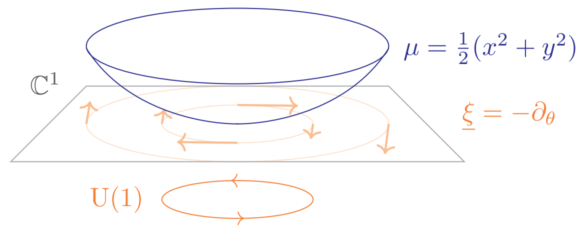
The action of scalar multiplication by the circle group $\mathrm{U}(1) = \{|z| = 1\}_{z\in\mathbb{C}}$ on $\mathbb{C}^{n+1}$ admits the moment map

$$\begin{aligned} \mu : \mathbb{C}^{n+1} &\longrightarrow \mathbb{R} \cong \mathfrak{u}(1) \\ z &\longmapsto \tfrac{1}{2}\|z\|^2 \end{aligned}$$

where $\|\cdot\|^2 = \sum_i x_i^2 + y_i^2$ is the usual norm squared on $\mathbb{C}^{n+1}$. When $\lambda > 0$, the action of $\mathrm{U}(1)$ on $\mathbb{C}^{n+1}$ on $\mu^{-1}(\lambda) = \{\|z\| = 1\}_{z\in C^{n+1}}$ is free and proper, and the reduced space is

$$\mathbb{C}P^{n+1}_\lambda = \{\|z\|^2 = 2\lambda\}/\mathrm{U}(1) \cong \mathbb{C}P^n$$

with reduced form ω_λ proportional to the Fubini–Study form $\omega_{FS} \in \Omega^2(\mathbb{C}P^n)$. Setting $n = 1$, we obtain the sphere $S^2 = \mathbb{C}P^1$ as a symplectic reduced space.



Example 50 (T^*Q). Consider the Hamiltonian action of G on $(T^*Q, -\mathrm{d}\theta)$ induced by a free and proper smooth action of G on Q. Recall that the canonical moment map $\mu : T^*Q \to \mathfrak{g}^*$ is given by

$$\mu_\xi(\alpha) = -\alpha(\underline{\xi}_q)$$

for $\alpha \in T_q^*Q$ and $\xi \in \mathfrak{g}$. The preimage of $0 \in \mathfrak{g}^*$ under μ is given by

$$\mu^{-1}(0) = \{\alpha(\underline{\mathfrak{g}}_Q) = 0\}_{\alpha \in T^*Q},$$

where $\mathfrak{g}_Q \subseteq TQ$ is the fundamental distribution of the action $G \curvearrowright Q$. It follows that

$$T^*Q_0 \;=\; \{\alpha(\underline{\mathfrak{g}}_Q) = 0\}/G \;\cong\; T^*(Q/G).$$

Example 51 ($\mathcal{O}_\lambda \subseteq \mathfrak{g}^*$). Let G be a Lie group and let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit in $\mathfrak{g}^*$. Recall that the canonical moment map for the coadjoint action $G \curvearrowright \mathcal{O}$ is the inclusion $\mu : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$. Fix $\lambda \in \mathfrak{g}^*$. If $\lambda \in \mathcal{O}$, then

$$\mu^{-1}(\lambda)/G_\lambda = \{\lambda\}/G_\lambda = \{\lambda\}$$

If $\lambda \notin \mathcal{O}$, then $\mu^{-1}(\lambda)$ is empty. The situation is more interesting when we consider the induced action of a subgroup $H \subseteq G$ on $\mathfrak{g}^*$.

Exercises

Consider a Hamiltonian system (M, ω, G, μ) with G-equivariant moment map $\mu : M \to \mathfrak{g}^*$, and suppose that $\lambda \in \mathfrak{g}^*$ is a regular value of μ.

1. *Reduction of dynamics.* Let $f \in C^\infty(M)$ be a smooth function which is preserved by the action of G. You may assume that if G preserves f, then there is a unique function $f_\lambda \in C^\infty(M_\lambda)$ such that

   $$i^* f = \pi^* f_\lambda$$

   where $i : \mu^{-1}(\lambda) \to M$ is the inclusion and $\pi^* : \mu^{-1}(\lambda) \to M_\lambda$ is the projection.

   i. Show that the Hamiltonian vector field $X_f \in \mathfrak{X}(M)$ is tangent to $\mu^{-1}(\lambda) \subseteq M$.

   ii. Prove that $\mathfrak{X}_f|_{\mu^{-1}(\lambda)}$ descends to M_λ and

   $$\pi_*(X_f|_{\mu^{-1}(\lambda)}) = X_{f_\lambda} \in \mathfrak{X}(M_\lambda).$$

2. If ω is exact, does it necessarily follow that $\omega_\lambda \in \Omega^2(M_\lambda)$ is exact? Prove or give a counterexample.

3. Show that the action of G on $\mu^{-1}(0) \subseteq M$ is locally free if and only if $0 \in \mathfrak{g}^*$ is a regular value of μ.

   Hint. Recall the identity $\omega(\underline{\xi}, X) = \langle \mu_* X, \xi \rangle$ and the nondegeneracy of ω. You may assume that the action of G on $\mu^{-1}(0)$ is locally free when $\xi \mapsto \underline{\xi}_x$ is an injection from $\mathfrak{g}$ to T_xM for all $x \in \mu^{-1}(0)$.

4. Confirm that $\mu : \mathbb{C}P^{n+1} \to \mathfrak{u}(1)$ is indeed a moment map for the action of $U(1) \curvearrowright (\mathbb{C}P^{n+1}, \omega)$ in Example 2.

5. Describe the diffeomorphism $\{\alpha(\underline{\mathfrak{g}}_Q) = 0\}_{\alpha \in T^*Q}/G \cong T^*(Q/G)$ in Example 3.

Chapter 5

Reduction by Stages

The symplectic reduction theorem takes a Hamiltonian manifold (M, ω, G, μ) and an admissible value $\lambda \in \mathfrak{g}^*$, and returns a symplectic manifold $(M_\lambda, \omega_\lambda)$.

$$(M, \omega, G, \mu) \xrightarrow{\text{reduce by } G} (M_\lambda, \omega_\lambda)$$

Heuristically, this process consists of encoding a collection of symmetries of (M, ω) in the form of a Lie group action $G \curvearrowright (M, \omega)$, and then removing these symmetries to obtain a reduced space $(M_\lambda, \omega_\lambda)$. We could, instead, remove the symmetries corresponding to a normal subgroup $N \subseteq G$. As we shall see, it is easy to show that $\nu : x \mapsto \mu(x)|_\mathfrak{n}$ is a moment map for this induced action. Taking an admissible value $\kappa \in \mathfrak{n}^*$, we may then reduce $(M, \omega, N, \mu|_\mathfrak{n})$ at an admissible value $\kappa \in \mathfrak{n}^*$ as usual. Let us assume, for simplicity, that $\kappa \in \mathfrak{n}^*$ is fixed by the action of N on $\mathfrak{n}^*$, so that $M_\kappa = \nu^{-1}(\kappa)/N$.

$$(M, \omega, G, \mu) \xrightarrow{\text{“forget” } G/N} (M, \omega, N, \mu|_\mathfrak{n}) \xrightarrow{\text{reduce by } N} (M_\kappa, \omega_\kappa)$$

We may wonder if the information inherent in G/N, which is lost in the transition to an action $N \curvearrowright (M, \omega)$, may be reclaimed in the form of an action of G/N on M_κ. This is at least plausible, since G/N and $\mu^{-1}(\kappa)/N$ each involve quotients by N. It turns out that, under certain broad conditions, we can. In this case, we may perform a second reduction, this time by the action of G/N.

$$(M, \omega, G, \mu) \xrightarrow{\text{reduce by } N} (M_\kappa, \omega_\kappa, G/N, \mu_\kappa - \lambda) \xrightarrow{\text{reduce by } G/N} (M_{\kappa,\tau},\, \omega_{\kappa,\tau})$$

We explain this notation in Theorem 56, below.

The perspective of what we will call *partial reduction* is that the reduction of a Hamiltonian manifold (M, ω, G, μ) is another Hamiltonian manifold $(M_\kappa, \omega_\kappa, G/N, \mu_\kappa - \lambda)$. Taking this view, we may consider the usual symplectic reduction of (M, ω, G, μ) as the Hamiltonian manifold $(M_\lambda, \omega_\lambda, 1, 0)$ incorporating a trivial action, reflecting the fact that we have “used up” the available symmetries. In this way, partial reduction is an extension of the process of symplectic reduction.

Reduction by stages is study of the twice-reduced space $(M_{\kappa,\tau},\, \omega_{\kappa,\tau})$, particularly in respect to a single, ordinary reduction $(M_\lambda, \omega_\lambda)$. The “stages” refer to the application of a partial reduction procedure. We will present a reduction by stages theorem in the simplest nontrivial case, in which $G = H \times N$ is a product of the subgroup N and the quotient $H = G/N$.

Key Points:

1. If (M, ω, G, μ) is a Hamiltonian manifold, and if $K \subseteq G$ is a Lie subgroup, then there is an induced Hamiltonian manifold $(M, \omega, K, \mu|_\mathfrak{k})$.

2. If $N \subseteq G$ is a closed central subgroup, then there is an induced Hamiltonian manifold $(M_\kappa, \omega_\kappa, G/N, \mu_\kappa - \lambda)$.

3. If $N \curvearrowright (M, \omega)$ and $H \curvearrowright (M, \omega)$ are commuting actions, then there is an induced action $H \times N \curvearrowright (M, \omega)$. Moreover, if ν and ρ are moment maps for the respective actions of N and H, then, subject to a technical compatibility condition, $\nu \oplus \rho$ is a moment map for the action of $H \times N$.

4. The reduction of a Hamiltonian manifold $(M, \omega, H \times N, \rho \oplus \nu)$ at $(\tau, \kappa) \in \mathfrak{h}^* \oplus \mathfrak{n}^*$ is symplectomorphic to the reduction of (M, ω, N, ν) at $\kappa \in \mathfrak{n}^*$, followed by the reduction by the induced action of H on M_κ. That is, $M_{(\kappa,\tau)} \cong (M_\kappa)_\tau$.

Remark. The conditions under which one may reduce by stages are quite general. The conditions appearing in this chapter, chosen to aid the exposition, are by comparison highly restrictive. The topic is considerably more complex in greater generality and the interested reader may wish to consult the literature for further details.

5.1 Subgroup Actions

In this brief section, we show that the property of being a Hamiltonian action $G \curvearrowright (M, \omega)$ descends to induced actions of subgroups $K \subseteq G$. We will make frequent use of this result.

Proposition 52. *Let (M, ω, G, μ) be a Hamiltonian manifold with G-equivariant moment map $\mu : M \to \mathfrak{g}^*$. If K is a Lie subgroup of G, then the function*

$$\begin{aligned} \nu : M &\to \mathfrak{k}^* \\ x &\mapsto \mu(x)|_{\mathfrak{k}} \end{aligned}$$

is a moment map for the action of K on (M, ω).

Proof. Since $\mu : M \to \mathfrak{g}^*$ is G-equivariant, it follows that $\nu : M \to \mathfrak{k}$ is K-equivariant. Restricting both sides of the equality $\mathrm{d}\mu_\xi = \iota_\xi \omega$ to $\mathfrak{k}$ for every $\xi \in \mathfrak{k}$ yields $\mathrm{d}\nu_\xi = \iota_\xi \omega$. □

5.2 Partial Reduction

Let (M, ω, G, μ) be a Hamiltonian manifold with G-equivariant moment map μ, and let $N \subseteq G$ be a closed normal subgroup. Our aim in this section is to reduce (M, ω, G, μ) by the action of N, to obtain a *partially reduced* system $(M_\kappa, \omega_\kappa, G/N, \mu_\kappa - \lambda)$. We explain our notation, and impose further conditions on N, as we proceed.

Observe, first of all, that the normality of N in G implies that $\mathfrak{n}^*$ is preserved by the coadjoint action of G on $\mathfrak{g}^*$. That is, there is a well-defined coadjoint action of G on $\mathfrak{n}^*$.

Lemma 53. *The assignment*

$$[g]_N \cdot [x]_N = [g \cdot x]_N, \qquad g \in G,\ x \in \nu^{-1}(\kappa),$$

defines a symplectic action of G/N on $(M_\kappa, \omega_\kappa)$.

Proof. We will show that the action is

i. well-defined,
ii. symplectic.

i. Fix $g, g' \in G$ and $x, x' \in \nu^{-1}(\kappa)$ so that $[g]_N = [g']_N$ and $[x]_N = [x']_N$. Thus

$$g' = mg, \qquad x' = nx,$$

for some $m, n \in N$, and hence

$$g' \cdot x' = mgn \cdot x.$$

By the normality of $N \subseteq G$, we have $n' = gng^{-1} \in N$. From

$$g' \cdot x' = mn'g \cdot x$$

we conclude that $[g \cdot x]_N = [g' \cdot x']_N$. Moreover, since $\nu : M \to \mathfrak{n}^*$ is G equivariant and G preserves κ, we have $[g \cdot x]_N \in M_\kappa$.

ii. This follows by the injectivity of $\pi^* : \Omega^*(M_\kappa) \to \Omega^*(\nu^{-1}(\kappa))$ and the equality

$$\pi^* g^* \omega_\kappa = g^* \pi^* \omega = i^* g^* \omega = i^* \omega = \pi^* \omega_\kappa, \qquad g \in G.$$

Here, as usual, we identify the group element $g \in G$ with its image in Diff M.

□

Definition 54. The *annihilator* of $\mathfrak{n} \subseteq \mathfrak{g}$ is the subspace

$$\mathfrak{n}^0 = \{\lambda \in \mathfrak{g}^* \mid \langle \lambda, \mathfrak{n} \rangle = 0\} \subseteq \mathfrak{g}^*.$$

Observe that each $\lambda \in \mathfrak{n}^0$ corresponds to a well-defined element $\bar{\lambda} \in (\mathfrak{g}/\mathfrak{n})^*$ given by the equality

$$\langle \bar{\lambda}, \xi + \mathfrak{n} \rangle_{\mathfrak{g}/\mathfrak{n}} = \langle \lambda, \xi \rangle_{\mathfrak{g}}, \qquad \xi \in \mathfrak{g}.$$

Lemma 55. *The linear map*

$$\begin{aligned} \mathfrak{n}^0 &\to (\mathfrak{g}/\mathfrak{n})^* \\ \lambda &\mapsto \bar{\lambda}, \end{aligned}$$

is an isomorphism of vector spaces.

Proof. The map $\lambda \mapsto \bar{\lambda}$ is injective since $\bar{\lambda} = 0$ implies $\langle \lambda, \xi \rangle_{\mathfrak{g}} = 0$ for all $\xi \in \mathfrak{g}$. The result follows as $\dim \mathfrak{n}^0 = \dim \mathfrak{g} - \dim \mathfrak{n} = \dim (\mathfrak{g}/\mathfrak{n})^*$. □

Recall that the subgroup $N \subseteq G$ is said to be *central* if $ng = gn$ for all $n \in N$ and $g \in G$.

Theorem 56. *Let (M, ω, G, μ) be a Hamiltonian manifold with G-equivariant moment map μ, let $\lambda \in \mathfrak{g}^*$ be G-invariant, let $N \subseteq G$ be a closed central subgroup, put $\kappa = \lambda|_{\mathfrak{n}} \in \mathfrak{n}^*$, let*

$$\begin{aligned} \nu : M &\to \mathfrak{n}^* \\ x &\mapsto \mu(x)|_{\mathfrak{n}} \end{aligned}$$

be the induced moment map for the action of N on (M, ω), and suppose that N acts freely on $\nu^{-1}(\kappa)$.

i. There is a unique function $\mu_\kappa : M_\kappa \to \mathfrak{g}^*$ such that $\pi^* \mu_\kappa = i^* \mu$, where $i : \nu^{-1}(\kappa) \to M$ is the inclusion and $\pi : \nu^{-1}(\kappa) \to M_\kappa$ is the projection.

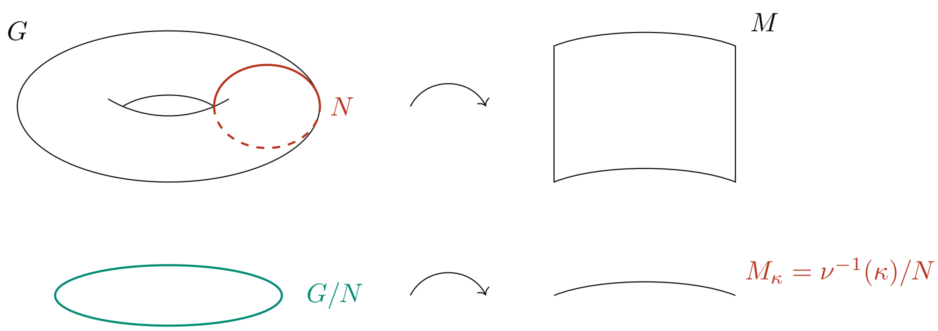
$$\begin{array}{cccc} & i^*\mu & & \mu \\ \pi^*\mu_\kappa & \nu^{-1}(\kappa) & \hookrightarrow & M \\ & \big\downarrow & & \\ \mu_\kappa & M_\kappa & & \end{array}$$

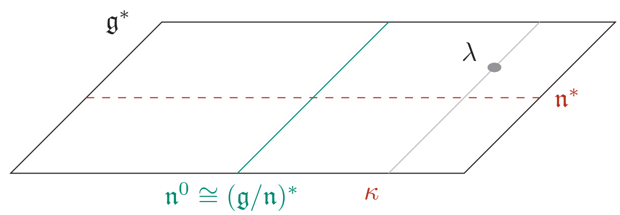
ii. *The map* $\mu_\kappa - \lambda : M_\kappa \to \mathfrak{g}^*$ *takes values in* $\mathfrak{n}^0 \cong (\mathfrak{g}/\mathfrak{n})^*$.

iii. *The function*

$$\mu_\kappa - \lambda : M \to (\mathfrak{g}/\mathfrak{n})^*$$

is a moment map for the action of G/N *on* $(M_\kappa, \omega_\kappa)$.





Proof. i. Since N acts freely on $\mu^{-1}(\kappa)$, it follows that $\kappa \in \mathfrak{n}^*$ is a regular value of $\nu : M \to \mathfrak{n}^*$. Consequently, the symplectic reduction theorem ensures that M_κ is a smooth manifold.

Let $n \in N$. Since $N \subseteq G$ is central, we have

$$\mathrm{Ad}_n \, \xi = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} n e^{t\xi} n^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{t\xi} = \xi$$

for all $\xi \in \mathfrak{g}$. It follows that $\mathrm{Ad}_n^* \lambda = \lambda$ for all $\lambda \in \mathfrak{g}^*$. In particular,

$$\mu(n \cdot x) = n \cdot \mu(x) = \mu(x)$$

for all $x \in M$. Thus $i^*\mu : \nu^{-1}(\kappa) \to \mathfrak{g}^*$ is N-invariant, and we conclude that $i^*\mu$ descends to a unique function $\mu_\kappa : \nu^{-1}(\kappa)/N \to \mathfrak{g}^*$.

ii. Fix $x \in \nu^{-1}(\kappa)$. We have $\mu_\kappa(\pi x)|_{\mathfrak{n}} = \nu(x) = \kappa$, from which $\mu_\kappa(\pi x) - \lambda\big|_{\mathfrak{n}} = 0$.

iii. From the G-equivariance of μ on M, we obtain the G/N-equivariance of μ_κ on M_κ. Since $\lambda \in \mathfrak{g}^*$ is fixed by the coadjoint action of G on $\mathfrak{g}^*$, it follows that $\mu_\kappa - \lambda : M \to \mathfrak{g}^*$ is G/N-equivariant.

Let $\chi \in \mathfrak{g}/\mathfrak{n}$ be represented by $\xi \in \mathfrak{g}$. Since $\mathfrak{n}$ acts trivially on $\nu^{-1}(\kappa)$, we obtain $i^*\mathrm{d}\mu_\xi = i^*\iota_\xi\omega$ and hence $\pi^*\mathrm{d}(\mu_\kappa)_\chi = \pi^*\iota_\chi\omega_\kappa$. By the injectivity of $\pi^* : \Omega^*(M_\kappa) \to \Omega^*(\nu^{-1}(\kappa))$, we conclude that

$$\mathrm{d}(\mu_\kappa - \lambda)_\chi = \mathrm{d}(\mu_\kappa)_\chi = \iota_\chi\omega_\kappa.$$

□

5.3 Product Groups

We now change our perspective and begin with the groups N and H, from which we construct a larger group $G = H \times N$. By contrast, in the preceding section, we began with an ambient group G and considered a subgroup N and quotient $H = G/N$.

Definition 57. Two smooth actions of Lie groups $N \curvearrowright M$ and $H \curvearrowright M$ are said to be *commuting actions* if

$$h \cdot (n \cdot x) = n \cdot (h \cdot x)$$

for all $h \in H$, $n \in N$, and $x \in M$. Locally, this implies $[\underline{\xi}, \underline{\eta}] = 0$ for $\xi \in \mathfrak{h}$ and $\eta \in \mathfrak{n}$.

Theorem 58. *Suppose that $N \curvearrowright (M, \omega)$ and $H \curvearrowright (M, \omega)$ are commuting Hamiltonian actions with respective moment maps $\nu : M \to \mathfrak{n}^*$ and $\rho : M \to \mathfrak{h}^*$, and suppose that ρ is N-invariant.*

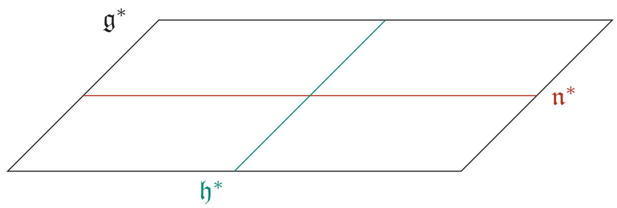
i. The induced action of $G = H \times N$ on (M, ω) is Hamiltonian, with moment map

$$\mu = \rho \oplus \nu : M \to \mathfrak{h}^* \oplus \mathfrak{n}^* \cong \mathfrak{g}^*.$$

ii. If $\kappa \in \mathfrak{n}^*$ and $\tau \in \mathfrak{h}^*$ are regular values of ν and ρ, respectively, if μ is G-equivariant, and if G acts freely on $\mu^{-1}(\tau, \kappa) \subseteq M$, then $H \curvearrowright (M, \omega)$ descends to a Hamiltonian action $H \curvearrowright (M_\kappa, \omega_\kappa)$ with induced moment map

$$\rho_\kappa : M_\kappa \to \mathfrak{h}^*,$$

and there is a natural symplectomorphism of reduced spaces $(M_\kappa)_\tau \cong M_{(\tau,\kappa)}$.



We prove this result through a series of lemmas.

Lemma 59. *There is a well-defined symplectic action of $G = H \times N$ on (M, ω), given by*

$$(h, n) \cdot x = h \cdot (n \cdot x).$$

The induced vector fields are

$$\begin{aligned} \mathfrak{h} \oplus \mathfrak{n} &\to \mathfrak{X}(M) \\ (\chi, \eta) &\mapsto \underline{\chi} + \underline{\eta}, \end{aligned}$$

where $\underline{\chi} \in \mathfrak{X}(M)$ is induced by the action of G, and $\underline{\eta} \in \mathfrak{X}(M)$ is induced by the action of N.

Proof. If (h, n) and $(h', n') \in H \times N$, then

$$\begin{aligned} [(h, n) \cdot_G (h', n')] \cdot x &= (hh', nn') \cdot x \\ &= h \underbrace{h'n}\, n'x \\ &= h \underbrace{nh'}\, n'x, && \text{since the actions of } N \text{ and } H \text{ commute,} \\ &= (h, n) \cdot [(h', n') \cdot x]. \end{aligned}$$

Thus, the action of $H \times N$ on M is well-defined. From

$$(h,n)^*\omega = h^*n^*\omega = \omega,$$

we deduce that the action of G is symplectic. We derive the identity $\underline{(\chi,\eta)} = \underline{\chi} + \underline{\eta}$ as follows,

$$\begin{aligned}
\underline{(\chi,\eta)}_x &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-t(\chi,\eta)} \cdot x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-t\chi}e^{-t\eta} \cdot x, && \text{since } N \text{ commutes with } H, \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-t\chi} \cdot x \;+\; \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-t\chi} \cdot x, && \text{by the Leibniz property, using } e^0 = 1, \\
&= \underline{\chi}_x + \underline{\eta}_x.
\end{aligned}$$

□

We are ready to prove part i. of Theorem 58.

Proof of Theorem 58, part i. Lemma 59 ensures that the action $G \curvearrowright (M,\omega)$ is symplectic. For all $\chi \in \mathfrak{h}$ and $\eta \in \mathfrak{n}$, we have

$$\mathrm{d}(\rho \oplus \nu)_{\chi+\eta} = \mathrm{d}\rho_\chi + \mathrm{d}\nu_\chi = \iota_\chi\omega + \iota_\eta\omega = \iota_{\chi+\eta}\omega.$$

Since,

$$\begin{aligned}
\{\rho_\chi, \rho_{\chi'}\} &= \rho_{[\chi,\chi']}, && \chi, \chi' \in \mathfrak{h} \\
\{\nu_\eta, \nu_{\eta'}\} &= \nu_{[\eta,\eta']}, && \eta, \eta' \in \mathfrak{n}
\end{aligned}$$

and

$$\{\nu_\eta, \rho_\chi\} = \mathcal{L}_\eta \rho_\chi = 0 = \mu_{[\eta,\chi]},$$

by the N-invariance of ρ, we deduce that the comoment map associated to $\mu = \rho \oplus \nu$ is a homomorphism of Lie algebras. □

We now turn to part ii. of Theorem 58.

Lemma 60. *If $\kappa \in \mathfrak{n}^*$ is a regular value of $\nu : M \to \mathfrak{n}^*$, then the action $H \curvearrowright (M,\omega)$ descends to an action $H \curvearrowright (M_\kappa, \omega_\kappa)$, with induced moment map $\rho_\kappa : M_\kappa \to \mathfrak{h}^*$.*

Proof. First observe that the N-invariant of ρ implies that $i^*\rho : \nu^{-1}(\kappa) \to \mathfrak{h}$ descends to M_κ. It remains to show that

i. H acts on $M_\kappa = \nu^{-1}(\kappa)/N$,
ii. ρ_κ is a moment map for $H \curvearrowright (M_\kappa, \omega_\kappa)$.

i. If $x \in \nu^{-1}(\kappa)$ and $h \in H$, then the $(H \times N)$-equivariance of $\rho \oplus \nu : M \to \mathfrak{h}^* \oplus \mathfrak{n}^*$ yields

$$\nu(h \cdot x) = h \cdot \nu(x) = \nu(x)$$

so that H preserves $\nu^{-1}(\kappa)$. Moreover, since N commutes with H, the action $H \curvearrowright M_\kappa$ given by

$$h \cdot [x]_N = [h \cdot x]_N$$

is well-defined.

ii. From the injectivity of $\pi^* : \Omega^*(M_\kappa) \to \Omega^*(\nu^{-1}(\kappa))$, and the equality

$$\pi^*\mathrm{d}(\rho_\kappa)_\chi = i^*\mathrm{d}\rho_\chi = i^*\iota_\chi\omega = \pi^*\iota_\chi\omega_\kappa,$$

we deduce that $\mathrm{d}(\rho_\kappa)_\chi = \iota_\chi\omega_\kappa$ for all $\chi \in \mathfrak{h}$. Similarly, the H-equivariance of ρ_κ follows from the H-equivariance of ρ.

□

Finally, we show that the reduced spaces $M_{(\tau,\kappa)}$ and $(M_\kappa)_\tau$ are canonically symplectomorphic.

Proof of Theorem 58, part ii. Fix $h \in H$ and $x \in \nu^{-1}(\kappa)$ such that $h \cdot [x]_N = [x]_N$. It follows that $h \cdot x = n \cdot x$ for some $n \in N$. Since $G = H \times N$ acts freely on $\mu^{-1}(\tau,\kappa) = \rho^{-1}(\tau) \cap \nu^{-1}(\kappa)$, and since $n^{-1}h \cdot x = x$, it follows that $h = n$ in $H \times N$, and we conclude that $h = 1$. Thus, the action of H on $\rho_\kappa^{-1}(\tau)$ is free, from which $\tau \in \mathfrak{h}^*$ is a regular value of $\rho_\kappa : M_\kappa \to \mathfrak{h}^*$, and hence $(M_\kappa)_\tau$ is a symplectic manifold.

Consider the diffeomorphism

$$\begin{aligned} \phi : M_{(\tau,\kappa)} = \big(\rho^{-1}(\tau) \cap \nu^{-1}(\kappa)\big)/H \times N &\xrightarrow{\sim} \rho_\kappa^{-1}(\tau)/H = (M_\kappa)_\tau \\ [x]_{H\times N} &\longmapsto \big[[x]_N\big]_H, \end{aligned}$$

where $x \in \rho^{-1}(\tau) \cap \nu^{-1}(\kappa)$, and consider the commutative diagram

$$\begin{array}{ccc} \rho^{-1}(\tau) \cap \nu^{-1}(\kappa) & \xrightarrow{\pi_N} & \rho_\kappa(\tau) \cap M_\kappa \\ \pi_{H\times N} \downarrow & & \downarrow \pi_H \\ M_{(\tau,\kappa)} & \xrightarrow[\phi]{\sim} & (M_\kappa)_\tau \end{array}$$

relating the quotient maps π_N, π_H, and $\pi_{H\times N}$ with the diffeomorphism ϕ. Since

$$\pi_{H\times N}^* \phi^* (\omega_\kappa)_\tau = \pi_H^* \pi_N^* (\omega_\kappa)_\tau = i^*\omega,$$

where $i : \rho^{-1}(\tau) \cap \nu^{-1}(\kappa) \to M$ is the inclusion, since

$$\pi_{H\times N}^* \omega_{(\tau,\kappa)} = i^*\omega,$$

and since π_N^*, π_H^*, and $\pi_{H\times N}^*$ are injective, we conclude that $\omega_{(\tau,\kappa)} = \phi^*(\omega_\kappa)_\tau$. That is, ϕ is a symplectomorphism. □

Review Exercises for Part I

1. Let V be a complex vector space equipped with a Hermitian inner product $(\,,) : V \times V \to \mathbb{C}$. Show that the real part

   $$\langle\,,\rangle = \mathrm{Re}\,(\,,) : V \times V \to \mathbb{R}$$

   defines a real-valued inner product on V, and that the imaginary part

   $$\omega = \mathrm{Im}\,(\,,) : V \times V \to \mathbb{R}$$

   defines a linear symplectic structure on the underlying real vector space V.

2. Fix $n \in \mathrm{N}$ and let $\mathbb{C}^{n\times n}$ denote the complex vector space of $n \times n$ complex matrices. Show that the pairing

   $$\begin{aligned} (\,,) : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} &\to \mathbb{C} \\ (A,B) &\mapsto \mathrm{tr}\, A^*B \end{aligned}$$

   is equal to the standard Hermitian inner product

   $$(A,B) = \sum_{i,j\le n} \bar{a}_{ij} b_{ij}.$$

   Conclude that $\langle\,,\rangle = \mathrm{Re}\,(\,,)$ defines an inner product on the underlying $2n^2$-dimensional vector space $\mathbb{C}^{n\times n}$.

3. *Unitary coadjoint orbits.* Fix $n \in \mathbb{N}$.

i. Recall that the space of *unitary matrices* $\mathrm{U}(n) \subseteq \mathbb{C}^{n \times n}$ consists of those matrices $U \in C^{n \times n}$ for which
$$UU^* = I$$
where $U^* = \bar{U}^T$ is the conjugate transpose of U, and where $I = \mathrm{diag}(1, \ldots, 1) \in \mathbb{C}^{n \times n}$ is the identity matrix. Differentiating a path $U_t : (-\epsilon, \epsilon) \to \mathrm{U}(n)$ with $U_0 = I$ yields
$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} U_t U_t^* = \dot{U}_0 U_0 + U_0 \dot{U}_0^* = \dot{U}_0 + \dot{U}_0^*,$$
where we have written $\dot{U}_0$ for $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} U_t$. It follows that the Lie algebra $\mathfrak{u}(n) = T_1 \mathrm{U}(n)$ consists of those matrices $A \in \mathbb{C}^{n \times n}$ for which
$$A + A^* = 0.$$
Such matrices are said to be *antihermitian* (or *skew-Hermitian*). Show that the adjoint action of $\mathrm{U}(n)$ on $\mathfrak{u}(n)$ is given by
$$\mathrm{Ad}_U B = UBU^*$$
for $U \in \mathrm{U}(n)$ and $B \in \mathfrak{u}(n)$.

Hint. The action of conjugation of $\mathrm{U}(n)$ on $\mathrm{U}(n)$ is given by $U \cdot V = UVU^{-1}$ for $U, V \in \mathrm{U}(n)$. Now let $V_t : (-\epsilon, \epsilon) \to \mathrm{U}(n)$ be a path satisfying $V_0 = I$ and $\dot{V}_0 = B$, and consider the induced path UV_tU^{-1}. What is the relation between U^* and U^T for the unitary matrix U?

ii. Show that the adjoint action of $\mathfrak{u}(n)$ on $\mathfrak{u}(n)$ is given by
$$\mathrm{ad}_A B = [A, B]$$
for $A, B \in \mathfrak{u}(n)$, where $[A, B] = AB - BA$ denotes the commutator.

Hint. Let $U_t : (-\epsilon, \epsilon) \to \mathrm{U}(n)$ be a path with $U_0 = I$ and $\dot{U}_0 = A$, and consider the induced path $\mathrm{Ad}_{U_t} B$.

iii. Show that
$$\langle A, B \rangle = -\mathrm{tr}\, AB$$
defines an Ad-invariant metric on $\mathfrak{u}(n)$. That is, show that
$$\langle \mathrm{Ad}_U A, \mathrm{Ad}_U B \rangle = \langle A, B \rangle$$
for all $U \in \mathrm{U}(n)$ and $A, B \in \mathfrak{u}(n)$.

Hint. To show that $\langle\, , \rangle$ is a metric, use part i. together with the fact that $A^* = -A$ for all $\mathfrak{u}(n)$. To show $\mathrm{U}(n)$-invariance, use the fact that $\mathrm{tr}\, XY = \mathrm{tr}\, YX$ for any matrices $X, Y \in \mathbb{C}^{n \times n}$, and choose X and Y carefully.

iv. Recall that a real vector space V may be identified with its dual V^* by means of a real-valued inner product $\langle\, , \rangle$ on V,
$$\begin{aligned} V &\xrightarrow{\sim} V^* \\ v &\mapsto \langle v, \rangle. \end{aligned}$$
Under this equivalence, we identify $A \in \mathfrak{u}(n)$ with the dual element $\langle A, \rangle \in \mathfrak{u}(n)^*$ given by
$$\langle A, B \rangle = -\mathrm{tr}\, AB, \qquad B \in \mathfrak{u}(n).$$

In particular, we identify $\mathfrak{u}(n)^*$ with the subspace of antihermitian matrices in $\mathbb{C}^{n\times n}$. Show that the coadjoint action of $\mathrm{U}(n)$ on $\mathfrak{u}(n)^*$ is given by

$$\mathrm{Ad}_U^* A = UAU^*$$

for all $U \in \mathrm{U}(n)$ and $A \in \mathfrak{u}(n)^*$. Conclude that the coadjoint orbit through $A \in \mathfrak{u}(n)^*$ is given by

$$\mathcal{O}_A = \{UAU^*\}_{U\in\mathrm{U}(n)} \subseteq \mathbb{C}^{n\times n}$$

with symplectic structure given by

$$\omega(\underline{B}_A, \underline{C}_A) = -\mathrm{tr}\,(A\,[B, C])$$

where $\underline{B}_A = -\mathrm{ad}_B A = -[B, A] \in T_A\mathcal{O}_A$, and likewise for $\underline{C}_A \in T_A\mathcal{O}_A$.

Hint. Recall that the defining condition for $\mathrm{Ad}_U A$ is that $\langle \mathrm{Ad}_U^* A, B\rangle = \langle A, \mathrm{Ad}_{U^{-1}} B\rangle$ for all $B \in \mathfrak{u}(n)$.

4. *Characterizing unitary coadjoint orbits.* Fix $n \in \mathbb{N}$ and $A \in \mathfrak{u}(n)^*$.

   i. Let $B \in \mathcal{O}_A$. Use the fact that conjugation preserves the spectrum to obtain $\mathrm{Spec}\,A = \mathrm{Spec}\,B$.

   ii. Suppose that $A = \mathrm{diag}(a_1, \ldots, a_n)$ and $B = \mathrm{diag}(b_1, \ldots, b_n) \in \mathfrak{u}(n)^*$ are diagonal. Prove that if $\mathrm{Spec}\,A = \mathrm{Spec}\,B$ then $B \in \mathcal{O}_A$.

   Hint. Observe that $\mathrm{Spec}\,A = \{a_1, \ldots, a_n\}$ and $\mathrm{Spec}\,B = \{b_1, \ldots, b_n\}$, so that A and B are equal up to permutation of diagonal entries. That is, there is a permutation σ on $\{1, \ldots, n\}$ such that $a_i = b_{\sigma(i)}$ for each $i \leq n$. Let $k, \ell \leq n$ and define the matrix $E_{k\ell} = (e_{ij})_{i,j} \in \mathbb{C}^{n\times n}$ by

   $$e_{ij} = \begin{cases} 1 & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise.} \end{cases}$$

   Define the matrix $P_{k\ell} = E_{k\ell} + E_{\ell k}$, and show that conjugation by P interchanges the entries a_k and a_ℓ in $A = \mathrm{diag}(a_1, \ldots, a_n)$. Now use the fact that every permutation on $\{1, \ldots, n\}$ is a product of transpositions.

   iii. Conclude that

   $$\mathcal{O}_A = \{B \in \mathfrak{u}^*(n) \,|\, \mathrm{Spec}\,A = \mathrm{Spec}\,B\}.$$

   Hint. Use the fact that every antihermitian matrix $A \in \mathfrak{u}(n)^*$ is diagonalizable by some unitary matrix $U \in \mathrm{U}(n)$.

Part II

Geometry of the Moment Map

Chapter 6

Morse Theory

If f is a smooth function on a compact manifold M, then we know that there must be critical points $x, y \in M$ at which f attains a minimum and a maximum. If f satisfies a mild nondegeneracy condition, then it turns out that much more can be said about the number and nature of the critical points of f. These properties are investigated in *Morse theory.*

In the next chapter, we will use the formalism of Morse–Bott functions in order to prove that the image of the moment map associated to a torus action $T \curvearrowright (M, \omega)$ is a convex polytope in $\mathfrak{t}^*$. This chapter presents the relevant background.

Key Points:

1. The *Hessian* of $f \in C^\infty(M)$ is well-defined as a symmetric bilinear form H_f on T_xM when x is a critical point of f. The Hessian H_f describes the second variation of f at x.

2. The *index* of a nondegenerate critical point x of f is the number of independent directions, emanating from x, in which f decreases.

3. The *Morse inequalities* relate the critical points of a Morse function $f \in C^\infty(M)$ to the topology of M. In particular, they relate the number of critical points of f of index $k \geq 0$ to the kth Betti number $b_k = \dim H_k(M)$.

Remark. Every homology group in this chapter should be understood to have coefficients in $\mathbb{R}$.

6.1 Bilinear Forms

We first review the language of bilinear forms. Suppose V is a vector space equipped with a bilinear form $B : V \times V \to \mathbb{R}$. Here we have in mind the tangent space $V = T_xM$ at a critical point x of a function $f \in C^\infty(M)$, together with the bilinear form given by the Hessian $B = H_f$.

Definition 61. We say that B is

i. *nondegenerate* if, for every $v \in V$, there is a $w \in V$ such that $B(v, w) \neq 0$,

ii. *positive* (resp. *negative*) *definite* if $B(v, v) \geq 0$ (resp. ≤ 0) for all $v \in V$, and $B(v, v) = 0$ only if $v = 0$.

The diagonalizability of symmetric matrices implies that there are subspaces $V_+, V_-, V_0 \subseteq V$ with

$$V = V_+ \oplus V_- \oplus V_0,$$

such that B is

i. *positive definite* on V_+,
ii. *negative definite* on V_-,
iii. *zero* on V_0.

Observe that B is nondegenerate precisely when $V_0 = 0$. While $V_0 \subseteq V$ is uniquely determined as the kernel of B, note that there are many available choices of V_+ and V_-.

Definition 62. The *index* of B is the dimension of any negative definite subspace V_-.

It is a fact of linear algebra, known as *Sylvester's law of inertia*, that the index of B is well-defined.

6.2 Critical Points and the Hessian

Let us review the basic elements of Morse theory. We begin with critical points and nondegeneracy, and finish with the definition of the Morse polynomial.

Definition 63. We call $x \in M$ a *critical point* of $f \in C^\infty(M)$ if $df = 0$ at x. In this case, we say that $f(x) \in \mathbb{R}$ is the *critical value* associated to x.

That is, x is critical for f if the derivative $Yf \in C^\infty(M)$ vanishes at x for every vector field $Y \in \mathfrak{X}(M)$.

Definition 64. The *Hessian* of $f \in C^\infty(M)$ at a critical point $x \in M$ is the symmetric bilinear form

$$\begin{aligned} Hf : T_xM \times T_xM &\longrightarrow \mathbb{R} \\ (X_x, Y_x) &\longmapsto XYf, \end{aligned}$$

where $X, Y \in \mathfrak{X}(M)$ are any vector fields extending $X_x, Y_x \in T_xM$, respectively.

Our first order of business is to prove that Hf is well-defined at critical points.

Lemma 65. *Let $x \in M$ be a critical point of $f \in C^\infty(M)$, and let $X, Y \in \mathfrak{X}(M)$ be arbitrary.*

i. $XYf = YXf$ at x.

ii. The Hessian $H_f : T_xM \times T_xM \to \mathbb{R}$ is well-defined at x.

Proof. i. We have

$$XYf - YXf = [X, Y]f = 0 \qquad \text{at } x.$$

ii. Choose any $X', Y' \in \mathfrak{X}(M)$ such that $X'_x = X_x$ and $Y'_x = Y_x$. We will show that

$$XYf \underset{(1)}{=} XY'f \underset{(2)}{=} X'Y'f \qquad \text{at } x.$$

To obtain Equality (1), observe that part i. implies

$$X(Y - Y')f = (Y - Y')Xf = 0 \qquad \text{at } x, \tag{$*$}$$

since $(Y - Y')_x \in T_xM$ is the zero vector. Equality (2) follows as $Xg = X'g$ at x for all $g \in C^\infty(M)$. □

Definition 66. Let M be a manifold and let $f \in C^\infty(M)$ be a smooth function on M.

i. The *index* of a critical point $x \in M$ is the index of the Hessian H_f as a bilinear form on T_xM.

ii. A critical point x of f is said to be *nondegenerate* if H_f is nondegenerate on T_xM.

iii. The function f is called a *Morse function* if every critical point of f is nondegenerate.

If x is a nondegenerate critical point of f, then the tangent space admits a splitting

$$T_xM = T_xM_+ \oplus T_xM_-,$$

such that the Hessian H_f is positive definite on T_xM_+ and negative definite on T_xM_-. Thus,

- f *increases* along the positive definite subspace T_xM_+,
- f *decreases* along the negative definite subspace T_xM_-.

Definition 67. Let M^n be a compact manifold and let f be a Morse function on M. The *Morse polynomial* of f is defined to be

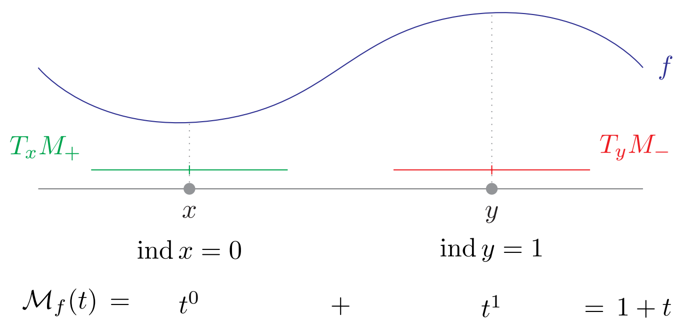
$$\mathcal{M}_f(t) = c_0 + c_1t + \cdots + c_nt^n \in \mathbb{Z}[t],$$

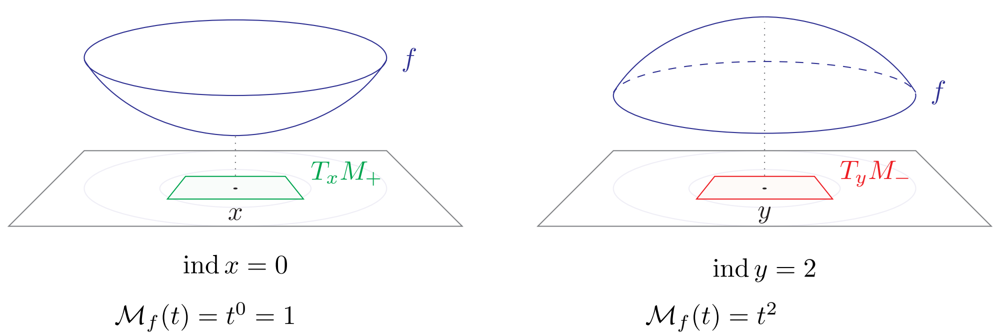
where $c_k \in \mathbb{N}$ is the number of critical points of f of index k.

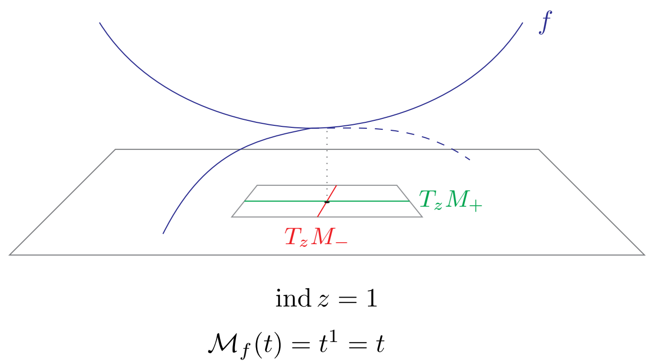
Equivalently, we have

$$\mathcal{M}_f(t) = \sum_{x\in C_f} t^{\operatorname{ind} x}$$

where $C_f \subseteq M$ is the set of critical points of f.







6.3 The Morse Inequalities

Having developed the language of Morse functions, we take the opportunity in this section to present the Morse inequalities. This is a truly fascinating result which relates the structure of the critical point set $C_f \subseteq M$ of a Morse function $f \in C^\infty(M)$ to the topology of the underlying manifold M. We remark that this section is logically independent of the remainder of the text.

Let M be a compact manifold.

Definition 68. The *Poincaré polynomial* of M is defined to be

$$\mathcal{P}(t) = b_0 + b_1 t + \cdots + b_n t^n \in \mathbb{Z}[t],$$

where $b_k = \dim H_k(M, \mathbb{R})$ is the kth Betti number of M.

Thus, $\mathcal{P}(M)$ encodes topological information about M. The Morse inequalities, which we now state, imply that $\mathcal{M}_f(t)$ also contains topological data.

Theorem 69 (Morse inequalities)**.** *If f is a Morse function on a compact manifold M, then*

$$\mathcal{M}_f(t) - \mathcal{P}(t) = (1+t)\, Q(t) \qquad (*)$$

for some polynomial $Q(t) = q_0 + q_1 t + \cdots + q_n t^n$ with non-negative coefficients $q_k \geq 0$.

Before we proceed to sketch the proof of this result, let us collect some immediate consequences.

Corollary 70. *If f is a Morse function on a compact manifold M, then*

i. f has at least b_k critical points of index k, for each $k \geq 0$,

ii. $\chi(M) = \mathcal{M}_f(-1)$, where $\chi(M) = \sum_{i \leq n} (-1)^i b_i$ is the Euler characteristic of M,

iii. (Morse lacunary principle) *if no two adjacent coefficients c_k, c_{k+1} are nonzero, then $\mathcal{M}_f(t) = \mathcal{P}(t)$.*

Proof. i. It follows from Theorem 69 that the coefficients of $\mathcal{M}_f(t) - \mathcal{P}(t)$ are nonactive. That is, $c_k \geq b_k$ for each $k \geq 0$.

ii. Evaluate each side of the Morse inequalities $(*)$ at $t = -1$.

iii. If $c_k = 0$ then $b_k = 0$ by part i. Now suppose $c_k > 0$. It follows that $c_{k-1} = c_{k+1} = 0$, and consequently that $b_{k-1} = b_{k+1} = 0$. Expanding $(1+t)q(t)$ yields

$$\begin{array}{rcccccccc} q(t) = & q_0 & + & q_1 t & + & \cdots & + & q_n t^n & \\ +\ t q(t) = & & + & q_0 t & + & \cdots & + & q_{n-1} t^n & + \ q_n t^{n+1} \\ \hline (1+t)\, q(t) = & q_0 & + & (q_0 + q_1) t & + & \cdots & + & (q_{n-1} + q_n) t^n & + \ q_n t^{n+1} \end{array}$$

Equating coefficients on either side of the Morse inequalities $(*)$, and using the fact that $q_i \geq 0$, we obtain

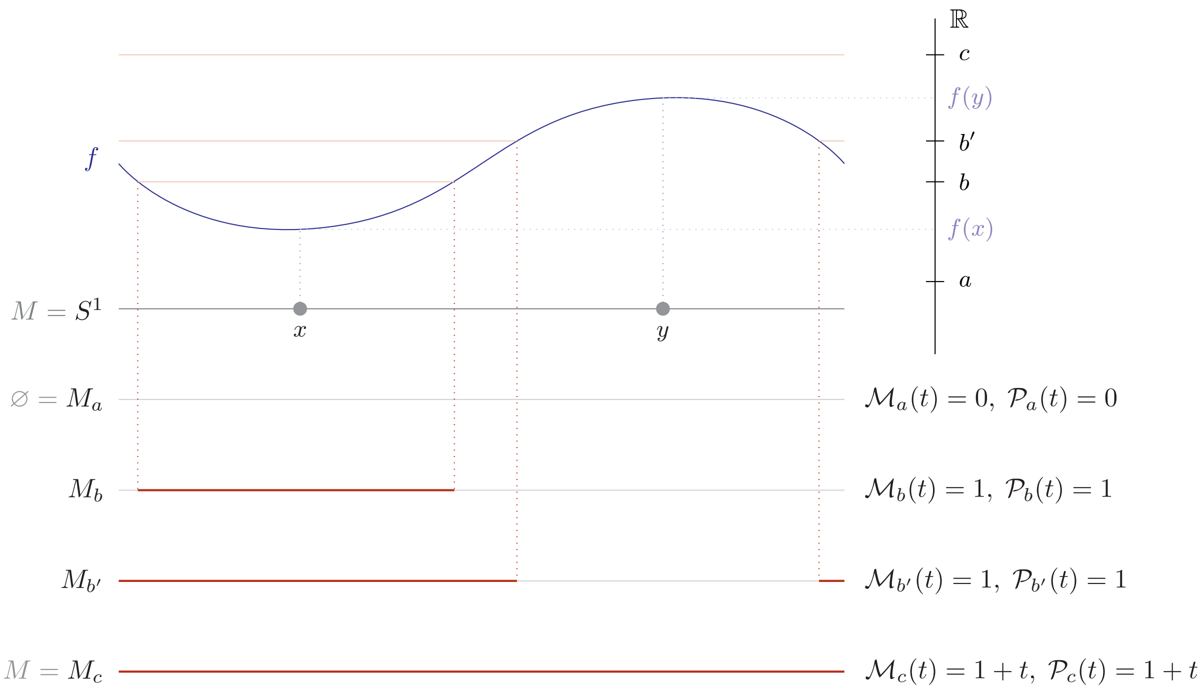
$$\left.\begin{array}{lrcll} t^{k+1}: & 0 & = & q_k + q_{k+1} & \Longrightarrow \quad q_k = 0 \\ t^k: & c_k - b_k & = & q_{k-1} + q_k & \\ t^{k-1}: & 0 & = & q_{k-1} + q_{k-1} & \Longrightarrow \quad q_{k-1} = 0 \end{array}\right\} \Longrightarrow q_{k-1} + q_k = 0,$$

and we conclude that $c_k = b_k$.

□

We now turn to the proof of the Morse inequalities. Our approach is to begin with the empty set $\varnothing \subseteq M$ and to use the Morse function f to "build" the manifold M in a controlled way. At each step, we confirm that the Morse inequalities hold.

First, we need to introduce some notation. For each $a \in \mathbb{R}$, define the open submanifold $M_a = f(x) < a_{x \in M}$, write $\mathcal{M}_a(t)$ for the Morse polynomial of M_a, and write $\mathcal{P}_a(t)$ for the Poincaré polynomial of M_a. Consider the illustration below, where we identify endpoints on the horizontal line to represent $M = S^1$.



Note the following:

- M_a is empty when a is less than the minimum value $f(x)$.
- M_c is the entire manifold when c is greater than the maximum value $f(y)$.
- The transition from M_a to M_b involves the addition of a nontrivial 0-cycle; the transition from M_b to M_c involves the addition of a nontrivial 1-cycle.
- M_b and $M_{b'}$ are homotopy equivalent.

With these observations in mind, we present the following key lemma.

Lemma 71. *Let f be a Morse function on a smooth manifold M and let $a < b$ be real numbers.*

A. If f has no critical values between a and b, then

$$\mathcal{P}_b(t) = \mathcal{P}_a(t).$$

B. If f has precisely one critical value $f(x)$ between a and b, then

$$\mathcal{P}_b(t) = \mathcal{P}_a(t) + t^{\operatorname{ind} x}$$

or

$$\mathcal{P}_b(t) = \mathcal{P}_a(t) - t^{\operatorname{ind} x - 1}.$$

Idea of proof. First observe that $M_a \subseteq M_b$.

A. The space M_a can be shown to be a deformation retract of M_b. In particular, M_b and M_a are homotopy equivalent, and thus $H_*(M_b) \cong H_*(M_a)$.

B. Put $k = \operatorname{ind} x$. By part A., it suffices establish this for a and b arbitrarily near $f(x) \in \mathbb{R}$. Homotopically, we obtain M_b by attaching a k-cell B_x to M_a,

$$M_b = M_a \cup B_x.$$

Let $\gamma \in H_{k-1}(M)$ be the $(k-1)$-cycle in M_a represented by ∂B_x. There are two possibilities:

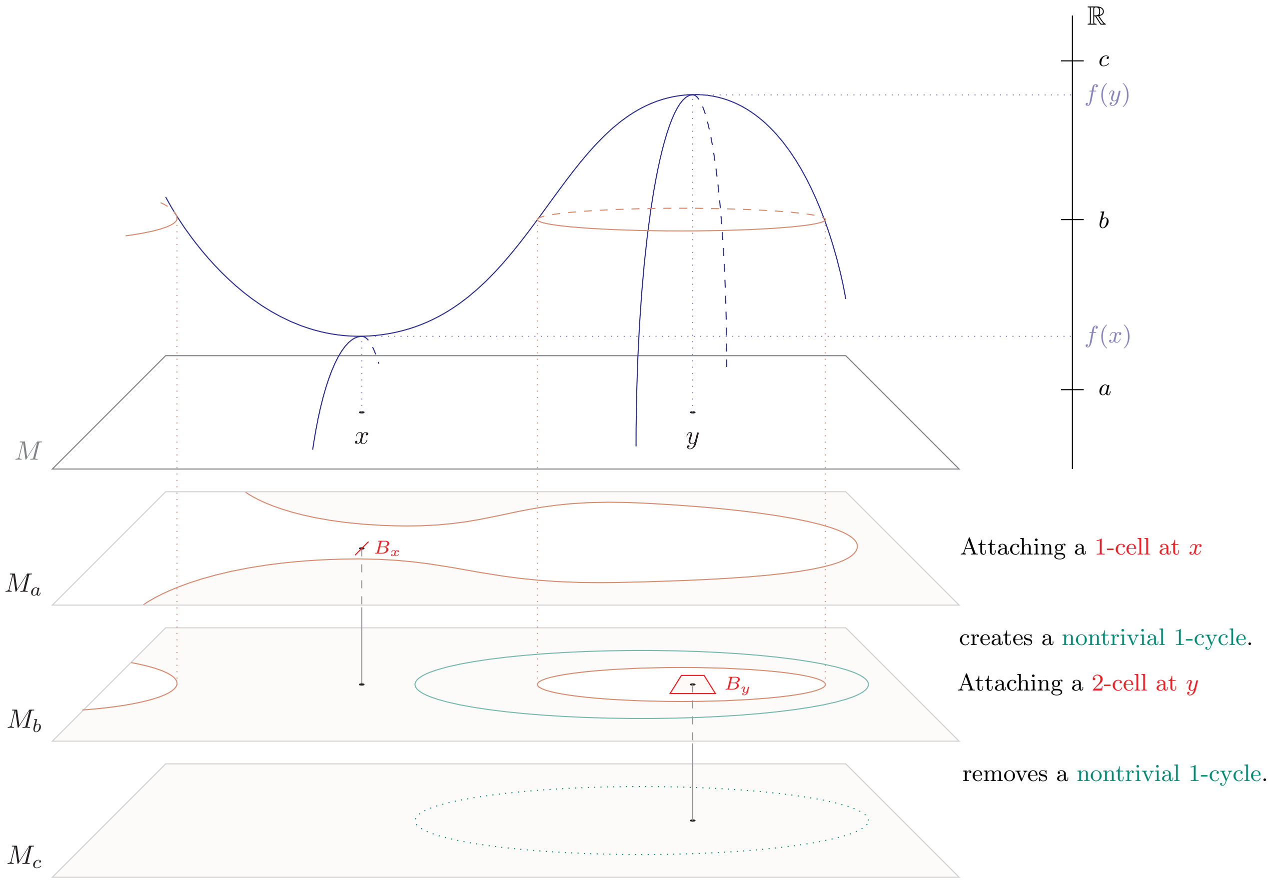
i. If ∂B_x bounds $\Sigma \subseteq M_a$, then $B_x \cup \Sigma$ represents a nontrivial k-cycle on M_b. Thus,

$$\mathcal{P}_b(t) = \mathcal{P}_a(t) + t^k.$$

ii. If ∂B_x represents a nontrivial homology class on M_a, then B_x trivializes ∂B_x on M_b. Consequently,

$$\mathcal{P}_b(t) = \mathcal{P}_a(t) + t^{k-1}.$$

$\square$



The Morse inequalities now follow easily.

Proof of the Morse inequalities. Let $\{x_i\}_{i \leq k} \subseteq M$ be the critical points of f. Let us suppose that the critical values $f(x_i) \in \mathbb{R}$ are distinct, and that the critical points are ordered so that

$$f(x_1) < f(x_2) < \cdots < f(x_k).$$

We may always perturb the function f to satisfy this condition, in such a way that preserves the critical points $\{x_i\}_{i\leq k}$ and their indices $\operatorname{ind} x_i$, so there is no loss of generality.

If $c \in \mathbb{R}$ is less than the minimum value of f, then $M_c = \varnothing$. In this case,

$$\mathcal{M}_a(t) = \mathcal{P}_a(t) = 0.$$

Now suppose that $f(x)$ is the unique critical value of f on the interval $(a, b) \subseteq \mathbb{R}$, and that

$$\mathcal{M}_a(t) - \mathcal{P}_a(t) = (1+t)\, Q_a(t)$$

for some $Q_a(t) \in \mathbb{Z}[t]$. Lemma 71 implies that

$$\mathcal{M}_b(t) = \mathcal{M}_a(t) + t^k$$
$$\mathcal{P}_b(t) = \mathcal{P}_a(t) + \begin{cases} t^k, \ \textit{or} \\ -t^{k-1} \end{cases}$$

where $k = \operatorname{ind} x$. Taking the difference yields

$$\begin{aligned} \mathcal{M}_b(t) - \mathcal{P}_b(t) &= \mathcal{M}_a(t) - \mathcal{P}_b(t) + t^k - \begin{cases} t^k \\ -t^{k-1} \end{cases} \\ &= (1+t)\, Q_a(t) + \begin{cases} 0 \\ (1+t)\, t^{k-1} \end{cases} \\ &= (1+t)\, Q_b(t). \end{aligned}$$

We proceed in this manner across every critical value of f. □

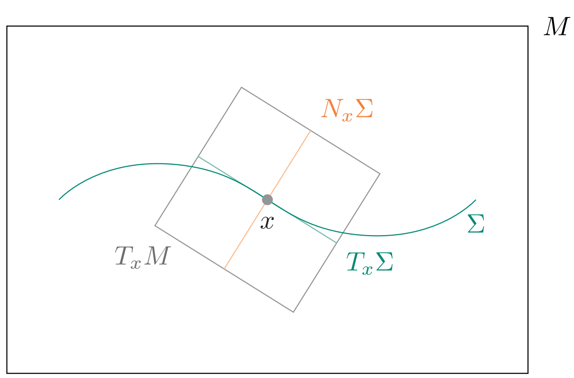
6.4 Bott–Morse Functions and Critical Manifolds

In this section, we extend the notion of a critical point $x \in M$ of a function $f \in C^\infty(M)$ to characterize a *critical submanifold* $\Sigma \subseteq M$ of f. In this new situation, the definition of a Morse function extends to that of a *Bott–Morse function*. We will invoke this material when we prove the convexity theorem in the next chapter.

Definition 72. A *normal bundle* for a submanifold $\Sigma \subseteq M$ is a vector subbundle of $N\Sigma \subseteq TM|_\Sigma \to \Sigma$ which satisfies

$$TM|_\Sigma = N\Sigma \oplus T\Sigma.$$

Here we identify $T\Sigma \to \Sigma$ as a subbundle of $TM|_\Sigma \to \Sigma$.



Definition 73. Fix a manifold M and a smooth function $f \in C^\infty(M)$.

i. If $\mathrm{d}f$ vanishes on a connected submanifold Σ, then we say that Σ is a *critical manifold* for f.

ii. A critical manifold $\Sigma \subseteq M$ of f is called *nondegenerate* if the Hessian H_f is nondegenerate as a bilinear form on every fiber of the normal bundle $N\Sigma$.

iii. The function f is called a *Bott–Morse function* if the critical set $C_f = \{\mathrm{d}f_x = 0\}_{x\in M}$ is equal to the union $\cup_\alpha \Sigma_\alpha$ of nondegenerate critical manifolds for f.

If $\Sigma \subseteq M$ is a nondegenerate critical manifold of f, then the fibers of the normal bundle admit a splitting

$$N_x\Sigma = N_x\Sigma_+ \oplus N_x\Sigma_-, \qquad x \in \Sigma,$$

such that H_f is positive definite on $N_x\Sigma_+$ and negative definite on $N_x\Sigma_-$. Since we assume our critical manifolds to be connected, it follows that this decomposition extends to the bundle $N\Sigma$. That is, $N\Sigma$ splits as the sum of positive and negative normal bundles

$$N\Sigma = N\Sigma_- \oplus N\Sigma_+.$$

Definition 74. The *index* of a nondegenerate critical manifold Σ of f is defined to be the rank of $N\Sigma_- \to \Sigma$.

Observe that, when $\Sigma = \{x\}$ consists of a single point in M, we have

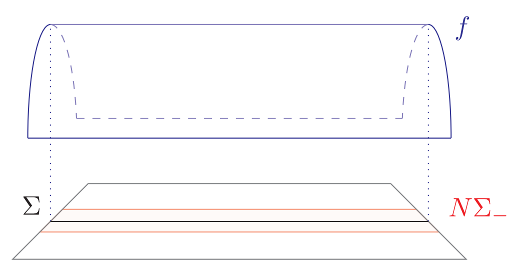
$$N_x\Sigma = T_xM, \qquad N_x\Sigma_+ = T_xM_+, \qquad N_x\Sigma_- = T_xM_-, \qquad T_x\Sigma = 0.$$

In particular, $\operatorname{ind}\Sigma = \operatorname{ind} x$.

If $\Sigma \subseteq M$ is a nondegenerate critical manifold of f, then

- f *increases* along the positive definite subbundle $N\Sigma_+$,
- f *decreases* along the negative definite subbundle $N\Sigma_-$.

$\operatorname{ind}\Sigma = 0$



$\operatorname{ind}\Sigma = 1$

Exercises

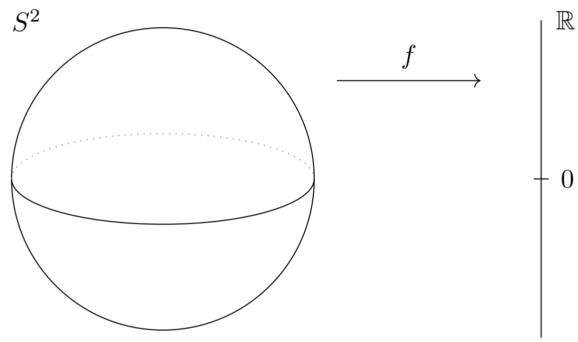
1. Give a new example (i.e. one that does not appear in the notes) of a manifold M and a function $f \in C^\infty(M)$ such that

   i. f is a Morse function,

   ii. f is not a Morse function, but is a Bott–Morse function,

   iii. f is neither a Morse function nor a Bott-Morse function.

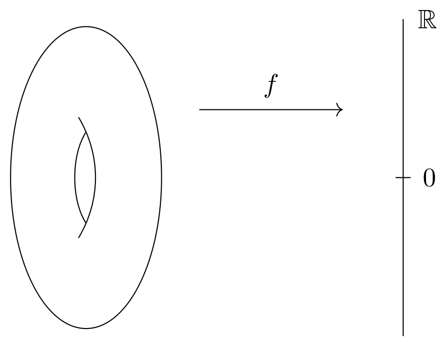
2. Consider the height function f on the sphere S^2.



i. Determine the Morse polynomial $\mathcal{M}_f(t)$ of f.

ii. Use part i. and the Morse lacunary principle to compute the homology of S^2.

3. Consider the indicated height function f on the torus T^2.



i. Determine the Morse polynomial $\mathcal{M}_f(t)$ of f.

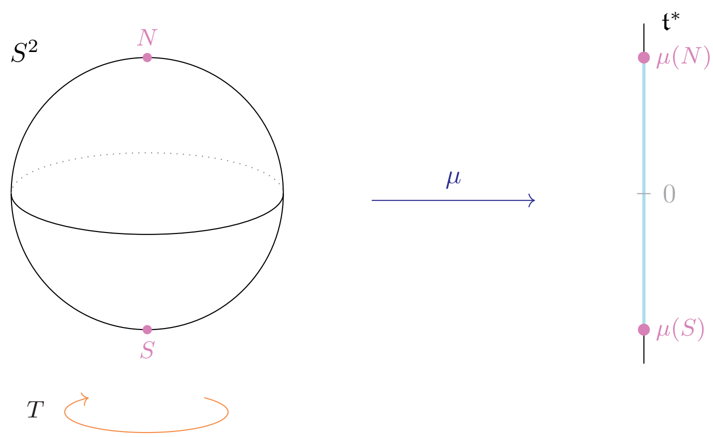
ii. Use part i. to compute the Euler characteristic $\chi(T^2)$.

4. Extend the previous exercise to obtain the Euler characteristic $\chi(\Sigma_g)$ of the g-holed torus Σ_g.

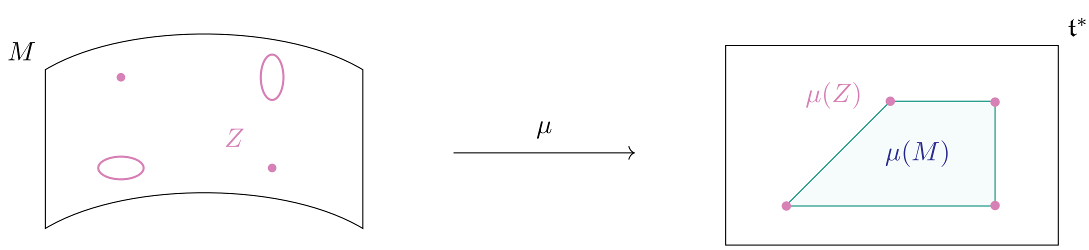
Chapter 7

The Moment Polytope

Consider the familiar action of the circle T on the sphere S^2 by rotations. The image of the standard moment map $\mu : S^1 \to \mathfrak{t}^*$ is the convex hull of the image of the north and south poles $N, S \in S^2$.

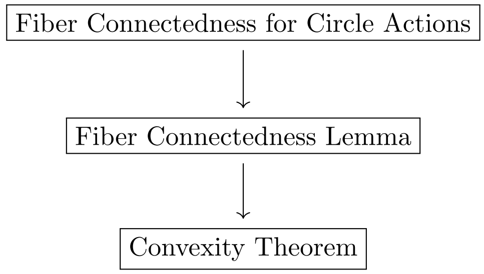


Note that N and S are the fixed points of the action of T on S^2. The *Atiyah–Guillemin–Sternberg convexity theorem* generalizes this observation to the class of Hamiltonian manifolds (M, ω, T, μ) for which M is compact and connected and T is a torus. In this case, $\mu(M) \subseteq \mathfrak{t}^*$ is the convex hull of the image under μ of the fixed-point set $Z \subseteq M$ of $T \curvearrowright M$. In particular, $\mu(M)$ is a convex polytope in $\mathfrak{t}^*$, called the *moment polytope* of the Hamiltonian manifold (M, ω, T, μ).



This is a landmark result in the theory of Hamiltonian manifolds. It was proved independently by Atiyah [1] and Guillemin–Sternberg [7]. Here we follow the proof of Atiyah.

The proof of the convexity theorem relies heavily on Lemma 76, which provides that the fibers of the moment map $\mu^{-1}(\lambda) \subseteq M$ are connected. This result, in turn, relies on Lemma 77, which establishes this connectedness in the special case in which the torus T is a circle.



Key Points:

1. If (M, ω, T, μ) is a Hamiltonian manifold with M compact and connected and T a torus, then the image $\Delta = \mu(M) \subseteq \mathfrak{t}^*$ is a convex polytope.
2. The polytope $\Delta \subseteq \mathfrak{t}^*$ is the convex hull of the image $\mu(Z) \subseteq \mathfrak{t}^*$ of the fixed points $Z \subseteq M$ of $T \curvearrowright M$.
3. If (M, ω, T, μ) is such that M is connected and T is a torus, then the preimage $\mu^{-1}(\lambda) \subseteq M$ is connected for every $\lambda \in \mathfrak{t}^*$.

7.1 The Convexity Theorem

In this section we state and prove the convexity theorem.

Theorem 75 (Atiyah–Guillemin–Sternberg convexity theorem)**.** *If (M, ω, T, μ) is a Hamiltonian manifold with M compact and connected and T a torus, then the image of the moment map $\mu(M) \subseteq \mathfrak{t}^*$ is a convex polytope.*

The proof follows from the following key lemma.

Lemma 76 (Fiber connectedness)**.** *If (M, ω, T, μ) is a Hamiltonian manifold with M connected and T a torus, then the level set $\mu^{-1}(\lambda) \subseteq M$ is connected for every $\lambda \in \mathfrak{t}^*$.*

We defer the proof to Section 7.2. For now, we take it on faith.

Proof of the convexity theorem. Let $Z \subseteq M$ be the fixed point set of $T \curvearrowright M$. We will show that

i. $\mu(M) \subseteq \mathfrak{t}^*$ is convex,
ii. if $\xi \in \mathfrak{t}$ generates T, then $\mu_\xi \in C^\infty(M)$ achieves its maximum value on Z,
iii. $\mu(M)$ is contained in the convex hull of $\mu(Z) \subseteq \mathfrak{t}^*$.

Parts i. and iii. together imply that $\mu(M)$ is equal to the convex hull of $\mu(Z) \subseteq \mathfrak{t}^*$. Since $\mathrm{d}\mu_\xi|_Z = 0$ implies that μ_ξ is constant on the connected components of Z, and since the number components of Z is finite by the compactness of M, we conclude that $\mu(Z)$ is a finite set and consequently that $\mu(M)$ is a convex polytope.

i. The subset $\mu(M) \subseteq \mathfrak{t}^*$ is convex precisely when:

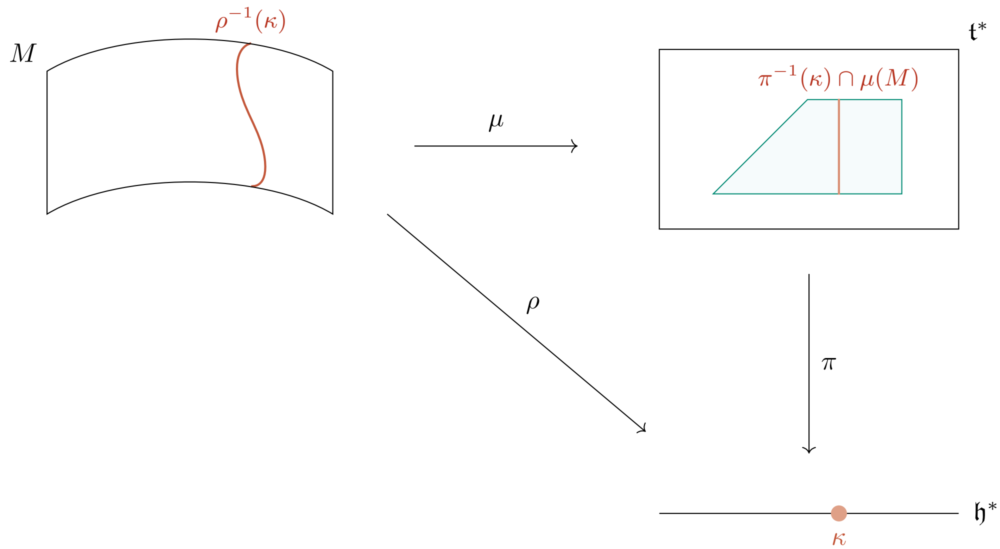
$$\ell \cap \mu(M) \text{ is connected for every affine linear subset } \ell \subseteq \mathfrak{t}^*.$$

Let $H \subseteq T$ be any subtorus of codimension 1, and let

$$\rho = \pi \circ \mu : M \to \mathfrak{h}^*$$

be the induced moment map for the action $H \curvearrowright (M, \omega)$. Here we write $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$ for the dual of the inclusion map $i : \mathfrak{h} \to \mathfrak{t}$. Observe that

- $\rho^{-1}(\kappa)$ is connected by Lemma 76,
- $\mu(\rho^{-1}(\kappa))$ is connected as it is the continuous image of a connected set,
- $\pi^{-1}(\kappa) \cap \mu(M)$ is connected since it is equal to $\mu(\rho^{-1}(\kappa))$.



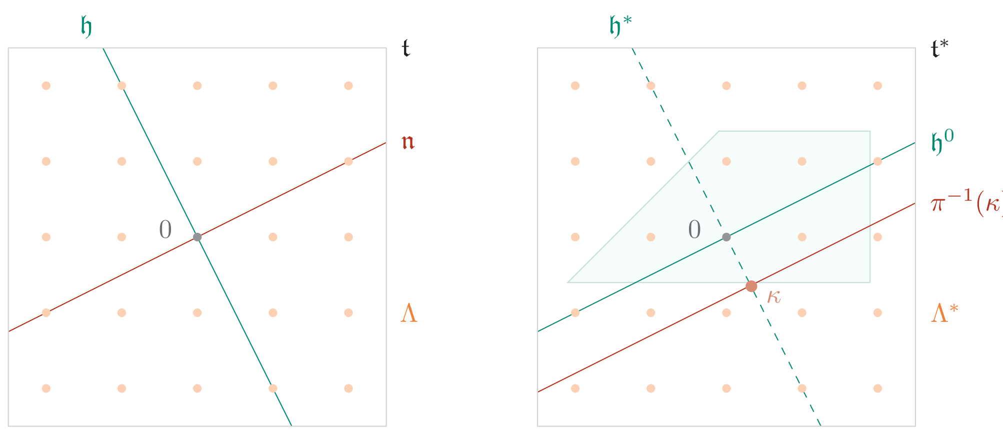
Our task is now to show that every line $\ell \subseteq \mathfrak{t}^*$ can be suitably approximated by $\pi^{-1}(\kappa)$, for some codimension 1 subtorus H and some $\kappa \in \mathfrak{h}^*$.

Let $\Lambda = \exp^{-1}(1_T)$ be the integral lattice in $\mathfrak{t}$. Any 1-dimensional subspace $\mathfrak{n} \subseteq \mathfrak{t}$, which meets Λ at any nonzero point, is tangent to a 1-dimensional subtorus $N = \exp(\mathfrak{n}) \subseteq T$. Moreover, there is a complementary codimension 1 subtorus $H \subseteq T$ with $H \times N = T$. The annihilator $\mathfrak{h}^0 = \{\langle \phi, \mathfrak{h} \rangle = 0\}_{\phi \in \mathfrak{t}^*}$ is a 1-dimensional subspace of $\mathfrak{t}^*$. As κ varies over $\mathfrak{h}^*$, the subsets $\pi^{-1}(\kappa) \subseteq \mathfrak{t}^*$ are precisely the translates of $\mathfrak{h}^0$.

Fix an line $\ell \subseteq \mathfrak{t}^*$. Since every 1-dimensional subspace $U \subseteq \mathfrak{t}^*$ meeting the dual lattice Λ^* at a nonzero point arises as $\mathfrak{h}^0$ for some choice of $\mathfrak{n} \subseteq \mathfrak{t}$, and since $\mu(M) \subseteq \mathfrak{t}^*$ is compact, it follows that there are codimension 1 subtori $H_i \subseteq T$ and elements $\kappa_i \in \mathfrak{h}_i$, such that

$$\pi_i^{-1}(\kappa_i) \cap \mu(M) \longrightarrow \ell \cap \mu(M), \qquad \text{as } i \to \infty,$$

in the sense that the maximum of the distance, with respect to any metric on $\mathfrak{t}^*$, from any point of $\pi_i^{-1}(\kappa_i) \cap \mu(M)$ to the set $\ell \cap \mu(M)$ approaches 0 as $i \to \infty$. Since $\pi_i^{-1}(\kappa_i) \cap \mu(M)$ is connected for each $i \geq 0$, and since $\mu(M)$ is closed, we conclude that $\ell \cap \mu(M)$ is connected.



ii. The set of critical points of $\mu_\xi \in C^\infty(M)$ is precisely the vanishing set of the associated Hamiltonian vector field $\underline{\xi} \in \mathfrak{X}(M)$. Now, since $\xi \in \mathfrak{t}$ generates T, the vanishing set of $\underline{\xi}$ is precisely the fixed point set of $T \curvearrowright M$. That is,

$$C_{\mu_\xi} = \{\underline{\xi}_x = 0\}_{x \in M} = Z.$$

In particular, μ_ξ attains its maximum at some point $z \in Z$.

iii. Recall that every $\phi \in (\mathfrak{t}^*)^*$ is of the form

$$\begin{aligned} \bar{\xi} : \mathfrak{t}^* &\to \mathbb{R} \\ \lambda &\mapsto \langle \lambda, \xi \rangle. \end{aligned}$$

for some $\xi \in \mathfrak{t}$. If ξ generates T, then part ii. guarantees a point $z \in Z$ with

$$\mu_\xi(M) \leq \mu_\xi(z).$$

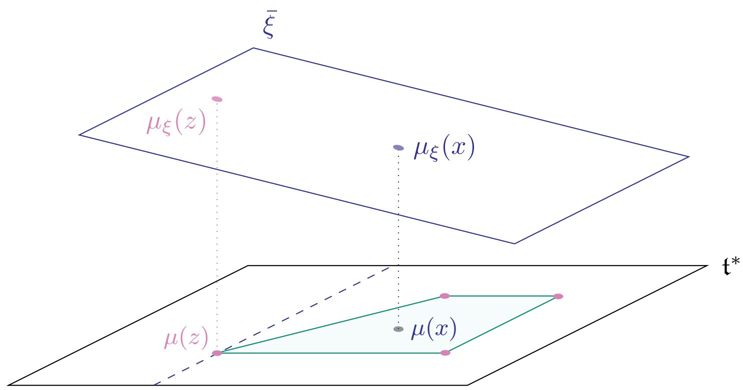
That is,

$$\bar{\xi}\big(\mu(M)\big) \leq \bar{\xi}\big(\mu(z)\big).$$

Thus, since the generic $\xi \in \mathfrak{t}$ generates T, it follows that the generic $\phi \in (\mathfrak{t}^*)^*$ satisfies

$$\phi\big(\mu(M)\big) \leq \phi(c)$$

for some $c \in \mu(Z)$. We conclude that $\mu(M)$ is contained in the convex hull of $\mu(Z)$.



□

7.2 Hamiltonian Circle Actions

To prove Lemma 76, we require a technical lemma.

Lemma 77. *If (M, ω, T, μ) is a Hamiltonian manifold with M connected and $T = S^1$, then*

i. $\mu_\xi \in C^\infty(M)$ is a Bott–Morse function for every $\xi \in \mathfrak{t}$,
ii. the index of each critical manifold $\Sigma \subseteq M$ of μ_ξ is even,
iii. the fiber $\mu^{-1}(\lambda)$ is connected for every $\lambda \in \mathfrak{t}^*$.

Sketch of proof. If $\xi = 0$, then the result is trivial. Thus suppose $\xi \neq 0$.

i. In general, the fixed point set of the action of a compact Lie group is an embedded submanifold. Let Σ be a connected component of the fixed-point manifold of $T \curvearrowright M$ and fix $x \in \Sigma$. Since $\underline{\xi}_x = 0$ for every $x \in \Sigma$, it follows that

$$\mathrm{d}\mu_\xi(X) = -\omega(\underline{\xi}_x, X) = 0,$$

for all $X \in TM|_\Sigma$. Thus, $\mathrm{d}\mu_\xi$ vanishes on Σ, from which it follows that Σ is a critical manifold for μ_ξ. Since T fixes x, there is an induced action of T on the tangent fiber

$$T_x M = T_x\Sigma \oplus N_x\Sigma.$$

As T acts by rotations on $N_x\Sigma$, there are symplectic coordinates $(x_i, y_i)_i$ on a neighborhood of x, in terms of which $\underline{\xi} = -\lambda_i \partial_{\theta_j} = \lambda_i(y\partial_x - x\partial_y)$ for some nonzero values $\lambda_i \in \mathbb{R}$. (This can be proved using the *equivariant Darboux theorem.*) In terms of these coordinates, we have

$$\mu_\xi = \frac{1}{2}\sum_i \lambda_i\,(x_i^2 + y_i^2), \tag{$*$}$$

and it follows that the Hessian H_{μ_ξ} is nondegenerate on $N_x\Sigma$.

ii. The index of Σ at x is the dimension of the subspace $N_x\Sigma_- \subseteq N_x\Sigma$ consisting of the directions in which μ_ξ is decreasing. In terms of Equation ($*$), this is equal to the number of coordinates x_i, y_i which are preceded by a negative coefficient $\lambda_i < 0$. In particular, the index is even.

iii. The essential fact for our purposes is that:

The index of each critical manifold Σ is odd.

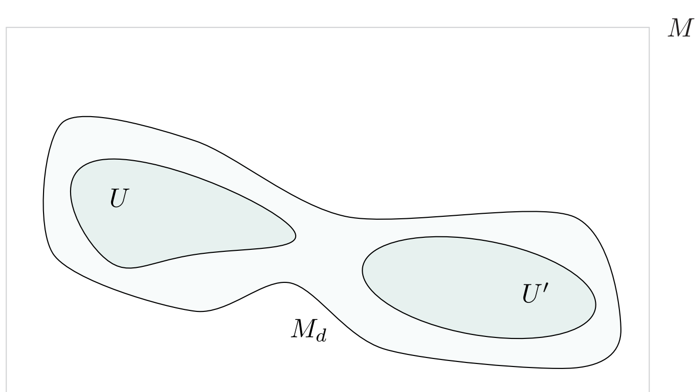
Recall from the previous chapter how we construct M by means of the manifolds

$$M_c = \{\mu_\xi(x) < c\}_{x \in M}$$

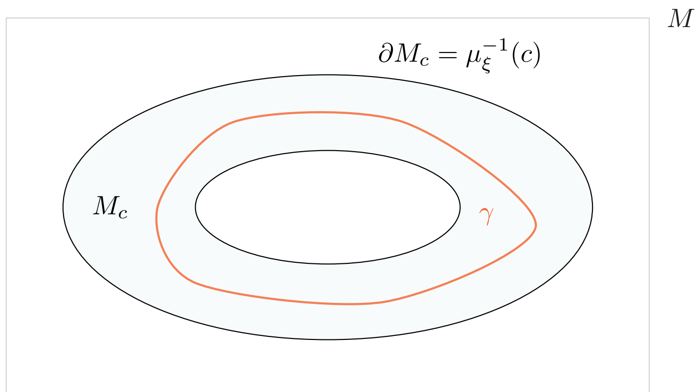
for $c \in \mathbb{R}$. Suppose $c \in \mathbb{R}$ is a regular value of μ_ξ. Suppose for a contradiction that $U, U' \subseteq M$ are two distinct components of M_c. Since

- M is connected,
- $\dim U = \dim U' = \dim M - 1$,

it follows that U and U' can only be joined in $M_d \supseteq M_c$ by passing through a critical value between c and d of index 1. This is the desired contradiction. We deduce that no such distinct components U and U' can exist. Therefore, M_c is connected for every regular value $c \in \mathbb{R}$, and thus for every $c \in \mathbb{R}$.



Suppose, again for a contradiction, that $\mu_\xi^{-1}(c) = \partial M_c$ is disconnected for some regular value $c \in \mathbb{R}$. It follows that any boundary component defines a nontrivial $(n-1)$-cycle in M_c, where $n = \dim M$. However, since $n-1$ is odd, and recalling our discussion from the previous chapter, a nontrivial $(n-1)$-cycle cannot be created during the construction of M from the intermediate manifolds M_a. From this contradiction we conclude that $\mu_\xi^{-1}(c)$ is connected.



□

We are now equipped to prove the fiber connectedness lemma.

Proof of Lemma 76. Since T is connected, it suffices to show that $M_\lambda = \mu^{-1}(\lambda)/T$ is connected.

If $T = H \times N$, where $H \subseteq T$ is a torus and $N \subseteq T$ is a circle, then

$$\mathfrak{t}^* = \mathfrak{h}^* \oplus \mathfrak{n}^*,$$
$$\mu = \rho \oplus \nu,$$
$$\lambda = (\chi, \kappa),$$

where $\rho : M \to \mathfrak{h}^*$ is a moment map for $H \curvearrowright (M, \omega)$ and $\nu : M \to \mathfrak{n}^*$ is a moment map for $N \curvearrowright (M, \omega)$.

Lemma 77 asserts that $\nu^{-1}(\kappa)$ is connected, and it follows that the partial reduced space $M_\kappa = \nu^{-1}(\kappa)/N$ is connected. Since $(M_\kappa, \omega_\kappa, H, \chi)$ is again a Hamiltonian manifold with H a torus, we may repeat this procedure until we obtain the full reduced space $(M_\lambda, \omega_\lambda)$. Since the partial reduced space is connected at every stage, we conclude that $M_\lambda = \mu^{-1}(\lambda)$ is connected. □

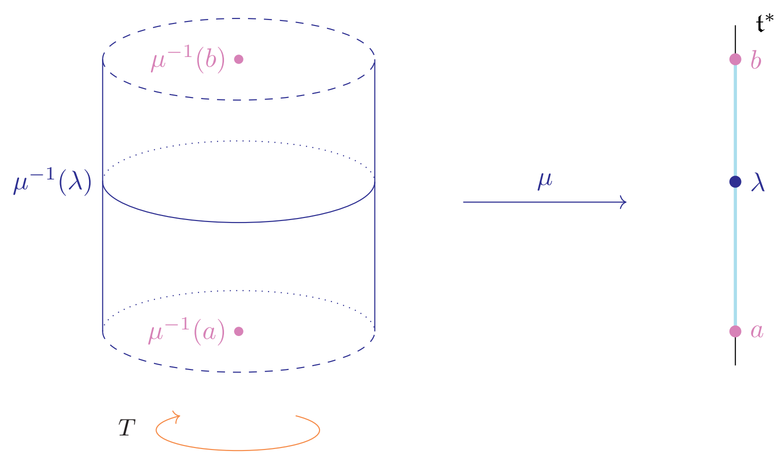
Chapter 8

Delzant's Theorem

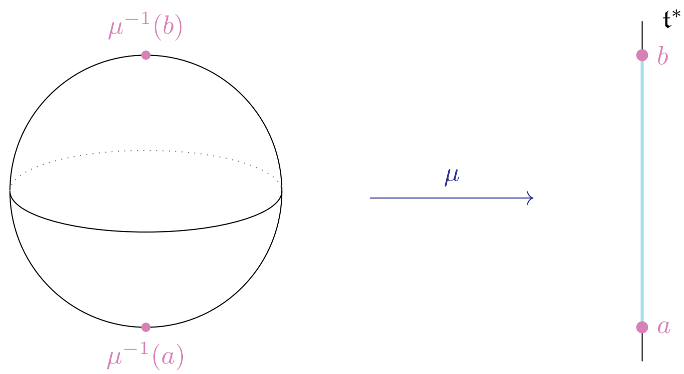
We have seen that, if (M, ω, T, μ) is a Hamiltonian manifold with M compact and connected and T a torus, then the image of the moment map $\mu(M)$ is a polytope in $\mathfrak{t}^*$. When the action $T \curvearrowright M$ is effective, and when the dimension of T is half the dimension of M, much more can be said.

Let us illustrate the underlying idea with an example. Suppose that the 1-torus $T = S^1$ acts effectively on a compact and connected 2-dimensional symplectic manifold (M, ω) with moment map $\mu : M \to \mathfrak{t}^*$. Moreover, suppose we know that the image of the moment map is $\mu(M) = [a, b]$ for some values $a, b \in \mathfrak{t}^*$. Our discussion above suggests that this data alone is enough to determine the symplectic manifold (M, ω). We know that

- $\mu^{-1}(\lambda)$ is a circle, for all $\lambda \in (a, b)$,
- $\mu^{-1}(a)$ and $\mu^{-1}(b)$ are points.



After a moment's reflection, we see that we have arrived again at the familiar example of a symplectic sphere.



It turns out that we can perform this reconstruction more generally, specifically in the setting of *symplectic toric manifolds*.

Definition 78. A *symplectic toric manifold* is a compact connected symplectic manifold (M^{2n}, ω), equipped with the effective action of a torus T^n of half the dimension of M.

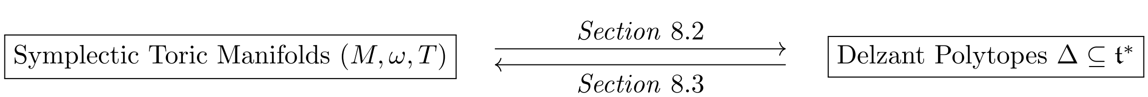
If $\mu : M \to \mathfrak{t}^*$ is a moment map for a symplectic toric manifold $T \curvearrowright (M, \omega)$, we will call (M, ω, T, μ) a *toric Hamiltonian manifold*.

In this case, the orbits of $T \curvearrowright M$ are precisely the fibers of $\mu : M \to \mathfrak{t}^*$. In particular, when $\lambda \in \mathfrak{t}^*$ is a regular value of $\mu : M \to \mathfrak{t}^*$, the fiber $\mu^{-1}(\lambda) \subseteq M$ is topologically an n-torus. At the other extreme, if $\lambda = \mu(x)$ is the image of a fixed point $x \in M$, then the fiber $\mu^{-1}(\lambda)$ is the singleton $\{x\} \subseteq M$. In general, if $\lambda \in \mathfrak{t}^*$ lies on a k-dimensional wall of the polytope $\mu(M) \subseteq \mathfrak{t}^*$, then the fiber $\mu^{-1}(\lambda)$ is a k-torus.

Of course, it is not at all clear that we can combine the preimages $\mu^{-1}(\lambda)$, as λ varies over some polytope $\Delta \subseteq \mathfrak{t}^*$, into a symplectic manifold (M, ω). Indeed, in general we cannot; the polytope $\Delta \subseteq \mathfrak{t}^*$ must be *Delzant*, a condition that we will define later on. In fact, *Delzant's theorem* establishes a 1–1 correspondence between toric Hamiltonian manifolds $(M^{2n}, \omega, T^n, \mu)$ and the Delzant polytopes $\Delta \in \mathfrak{t}^*$.

Theorem 79 (Delzant)**.** *If (M, ω, T, ω) is a toric Hamiltonian manifold, then the image of the moment map $\mu(M) \subseteq \mathfrak{t}^*$ is a Delzant polytope. Moreover, every Delzant polytope $\Delta \subseteq \mathfrak{t}^*$ is the moment polytope for some toric Hamiltonian manifold (M, ω, T, μ).*

This theorem was originally established in [5]. The remainder of this chapter is dedicated to the proof.



Key Points:

1. A $2n$-dimensional symplectic manifold (M^{2n}, ω) equipped with the effective action of a n-torus T^n is called a *symplectic toric manifold*.

2. *Delzant's theorem* asserts that the assignment of moment polytopes $\mu(M) \subseteq \mathfrak{t}^*$ describes a 1–1 correspondence between symplectic toric manifolds (M, ω, T), up to equivariant symplectomorphism, and Delzant polytopes $\Delta = \mu(M) \subseteq \mathfrak{t}^*$, up to translation.

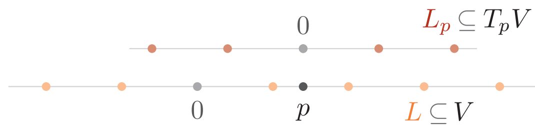
3. Every symplectic toric manifold arises as the partial reduction of the $2d$-dimensional *universal symplectic toric manifold* $(\mathbb{C}^d, \omega, T^d, \mu)$, for some $d \geq 1$.

8.1 Delzant Polytopes

Our first task is to define the class of Delzant polytopes $\Delta \subseteq \mathfrak{t}^*$. As guaranteed by Delzant's theorem, these are the polytopes that arise as the image of a moment map $\mu(M) \subseteq \mathfrak{t}^*$ associated to a symplectic toric manifold $T \curvearrowright (M, \omega)$.

Consider a vector space V and a full sublattice $L \subseteq V$. For concreteness, we may take $V = \mathbb{R}^k$ and $L = \mathbb{Z}^k$. However, our interest in this chapter regards $V = \mathfrak{t}^*$ and $L = \Lambda^*$, the dual lattice to the integral lattice $\Lambda = \exp^{-1}(1_T)$.

Recall that the tangent space to V at any point $p \in V$ is canonically linearly isomorphic to V. The image of $L \subseteq V$ under this identification $V \xrightarrow{\sim} T_pV$ is a lattice in T_pV, which we will denote by $L_p \subseteq T_pV$.



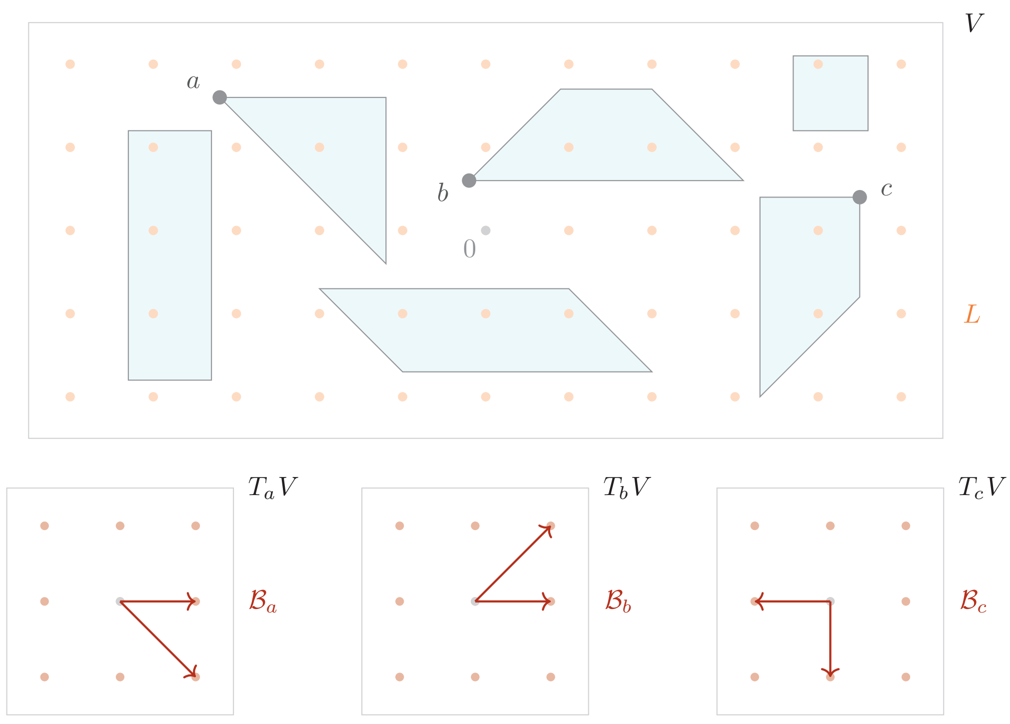
We recall the following definition.

Definition 80. A *basis* of a lattice $L \subseteq V$ is a minimal generating set $\mathcal{B} \subseteq L$ of L.

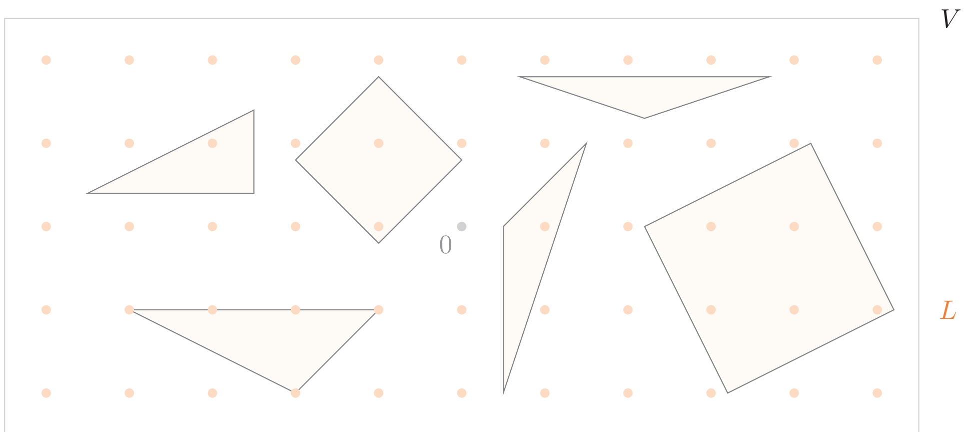
The primary new construction of this chapter is the following.

Definition 81. A *Delzant polytope* in V is a convex polytope $\Delta \subseteq V$ such that the edges of Δ at each vertex $v \in \Delta$ are tangent to a basis $\mathcal{B}_v$ of L_v.

Equivalently, the tangent cone to Δ at v is spanned by $\mathcal{B}_v$. Here are some examples of Delzant polytopes:



And here are some non-examples:



Let T be a torus. Recall that the *integral lattice* $\Lambda \subseteq \mathfrak{t}$ is defined to be the kernel of the exponential map $\exp : \mathfrak{t} \to T$, that is,

$$\Lambda = \{\xi \in \mathfrak{t} \,|\, \exp(\xi) = 1_T\} = \exp^{-1}(1_T),$$

and that the *lattice of integral forms* $\Lambda^* \subseteq \mathfrak{t}^*$ is the dual lattice

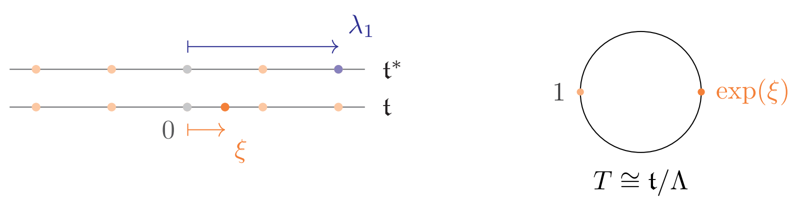
$$\Lambda^* = \{\lambda \in \mathfrak{t}^* \,|\, \lambda(\Lambda) \subseteq \mathbb{Z}\}.$$

8.2 Moment Polytopes of Symplectic Toric Manifolds

The aim of this section is to establish that the moment polytope $\Delta = \mu(M)$ of a toric Hamiltonian manifold (M, ω, T, μ) is Delzant. Our approach is to appeal to the theory of torus actions on a vector space V, and then apply the equivariant Darboux theorem.

Lemma 82. *If the n-dimensional torus T^n acts effectively on $\mathbb{C}^n$, then the weights of $T^n \curvearrowright \mathbb{C}^n$ form a basis $\mathcal{B}$ of the lattice of integral forms $\Lambda^* \subseteq \mathfrak{t}^*$.*

Proof. Suppose for a contradiction that $\mathcal{B} = \{\lambda_i\}_i \subseteq \Lambda$ does not generate the lattice Λ^*. It follows that there is a $\xi \notin \Lambda$ such that $\langle \lambda_i, \xi \rangle \in \mathbb{Z}$ for all $\lambda_i \in \mathcal{B}$. Consequently, $\exp(\xi) \in T$ is a nontrivial element of T that acts trivially on V. This contradicts the assumption that the action $T \curvearrowright V$ is effective. □



We now apply

Proposition 83. *If (M, ω, T, μ) is a toric Hamiltonian manifold with M, then the moment polytope $\mu(M) \subseteq \mathfrak{t}^*$ is Delzant.*

Proof. Since the action $T \curvearrowright M$ is effective, and since T is abelian, it follows that the principal orbit type of $T \curvearrowright M$ is (1_T). In particular, the action of T on M is free on a dense open subset $M_{(1)} \subseteq M$. Thus, if $x \in M$ is fixed by T, then the action $T \curvearrowright T_xM$ is effective, and Lemma 82 implies that the weights of $T \curvearrowright T_xM$ form a basis of Λ^*. The result follows since we have seen, in the proof of the fiber connectedness lemma, that the weights of $T \curvearrowright T_xM$ are tangent to the vertices of $\mu(M)$ which meet $\mu(x)$.

In our proof of the fiber connectedness lemma, we described that □

8.3 The Universal Symplectic Toric Manifold

In this section, we show that every Delzant polytope is realized as the moment polytope of some toric Hamiltonian manifold.

Put $d = n + 1$. Let the d-torus $T^d \subseteq \mathbb{C}^d$ act on $\mathbb{C}^d$ by

$$(t_1, \dots, t_d) \cdot (z_1, \dots, z_d) = (t_1 z_1, \dots, t_d z_d).$$

A moment map for the action $T \curvearrowright \mathbb{C}^d$ is given by

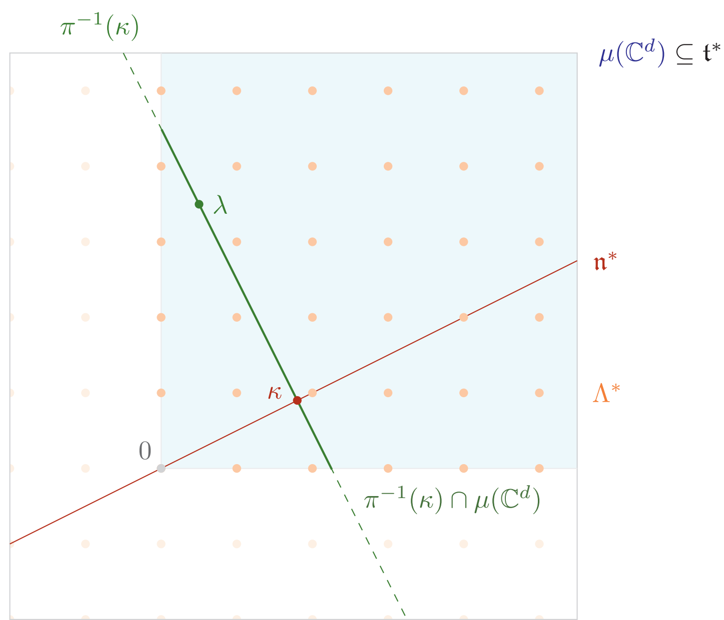
$$\mu(z_1, \dots, z_d) = \frac{1}{2}\left(|z_1|^2, \dots, |z_d|^2\right)$$

where we identify $\mathfrak{t}^*$ with $\mathbb{R}^d$. Note that the image of μ is identified with the positive octant in $\mathbb{R}^d$. We call $(\mathbb{C}^d, \omega, T, \mu)$ the *universal toric symplectic manifold* because, as we shall see, every n-dimensional toric symplectic manifold arises as the partial reduction of $(\mathbb{C}^d, \omega, T, \mu)$ by a circle subgroup $N \subseteq T$. Notwithstanding our terminology, note that $(\mathbb{C}^d, \omega, T)$ is not, in fact, a toric symplectic manifold, as it is noncompact.

The partial reduction of $(\mathbb{C}^d, \omega, T^d, \mu)$ by any circle subgroup $N \subseteq T^d$ at the level $\kappa \in \mathfrak{n}^*$ is the Hamiltonian manifold $(\mathbb{C}^d_\kappa, \omega_\kappa, H, \mu_\kappa)$, where

$$\mathbb{C}^d_\kappa = \nu^{-1}(\kappa)/N,$$

and where $\nu = \pi \circ \mu : M \to \mathfrak{n}^*$ is the induced moment map for the action $N \curvearrowright (M, \omega)$.



Our interest in this construction is expressed by the following fact of polytopes, which we state without proof.

Lemma 84. *Every n-dimensional Delzant polytope is of the form $\Delta = \pi^{-1}(\kappa) \cap \mu(\mathbb{C}^d)$.*

Our task now is to show that $\Delta = \pi^{-1}(\kappa) \cap \mu(\mathbb{C}^d)$ is the moment polytope of $(M_\kappa, \omega_\kappa, H, \mu_\kappa)$, and that this Hamiltonian manifold is toric.

Fix $\lambda \in \mu(\mathbb{C}^d)$. Under the identification

$$\begin{aligned} \mathfrak{t}^* &\longleftrightarrow \mathbb{R}^d \\ \lambda &\qquad (\lambda_1, \dots, \lambda_d) \end{aligned}$$

we have

$$\mu^{-1}(\lambda) = S\big(\sqrt{\lambda_1}\big) \times \cdots \times S\big(\sqrt{\lambda_d}\big) \subseteq \mathbb{C}^d,$$

where $S_k = S\big(\sqrt{\lambda_k}\big)$ denotes the circle of radius $\sqrt{\lambda_k}$ in $\mathbb{C}$. Moreover, the torus $T \cong \mathrm{U}(1)^d$ acts on the fiber $\mu^{-1}(\lambda) = S_1 \times \cdots \times S_k$ by coordinatewise rotations on each factor. If λ is an interior point in $\mu(M)$, then $\lambda_k > 0$ for each $k \leq d$, and the fiber $\mu^{-1}(\lambda)$ is diffeomorphic to a d-torus. If, on the other hand, λ lies on an ℓ-dimensional wall of $\mu(M)$, then only ℓ of the parameters λ_k are nonzero, and the fiber $\mu^{-1}(\lambda)$ is diffeomorphic to an ℓ-torus.

As noted in the proof of the convexity theorem, we have

$$\pi^{-1}(\kappa) \cap \mu(M) = \mu(\nu^{-1}(\kappa)). \tag{$*$}$$

Thus,

$$\begin{aligned}
\mu_\kappa(\mathbb{C}^d_\kappa) &= \mu_\kappa\big(\nu^{-1}(\kappa)/N\big), && \text{by the definition of } \mathbb{C}^d_\kappa,\\
&= \mu(\nu^{-1}(\kappa)), && \text{since } \mu_\kappa\big([x]_N\big) = \mu(x) \text{ for all } x \in \nu^{-1}(\kappa),\\
&= \pi^{-1}(\kappa) \cap \mu(M), && \text{by Equation } (*).
\end{aligned}$$

Lemma 85. *If $\pi^{-1}(\kappa) \subseteq \mathfrak{t}^*$ meets the interior of $\mu(\mathbb{C}^d)$, then $H \curvearrowright (\mathbb{C}^d_\kappa, \omega_\kappa)$ is a symplectic toric manifold.*

Proof. Fix a $\lambda \in \pi^{-1}(\kappa)$ which lies in the interior of $\mu(\mathbb{C}^d)$. It follows that λ is a regular value of $\mu : M \to \mathfrak{t}^*$ and consequently that κ is a regular value of $\nu : M \to \mathfrak{n}^*$. We will show that

i. M_κ is a smooth manifold,

ii. $\dim H = \frac{1}{2} \dim \mathbb{C}^d_\kappa$,

iii. $H \curvearrowright (\mathbb{C}^d_\kappa, \omega_\kappa)$ is effective.

This establishes that $H \curvearrowright (\mathbb{C}^d_\kappa, \omega_\kappa)$ is a symplectic toric manifold.

i. Since $\kappa \in \mathfrak{n}^*$ is a regular value of $\nu : M \to \mathfrak{n}^*$, the preimage $\nu^{-1}(\kappa) \subseteq \mathbb{C}^d$ is smooth. Note that N acts freely on the complement of the origin 0 in $\mathbb{C}^d$. Now, since $\pi^{-1}(\kappa)$ meets the interior of $\mu(M)$, the preimage $\pi^{-1}(\kappa)$ cannot contain the origin $0 \in \mathbb{C}^d$. Therefore, N acts freely on the smooth manifold $\pi^{-1}(\kappa)$, and therefore the quotient $\mathbb{C}^d_\kappa = \nu^{-1}(\kappa)/N$ is smooth.

ii. From

$$\dim T = d, \qquad \dim N = 1, \qquad T = H \times N,$$

we deduce that $\dim H = d - 1$. Since $\lambda \in \mathfrak{t}^*$ is a regular value of $\mu : \mathbb{C}^d \to \mathfrak{t}^*$, it follows that $\kappa \in \mathfrak{n}^*$ is a regular value of $\nu : M \to \mathfrak{n}^*$. Thus,

$$\dim\big(\nu^{-1}(\kappa)/N\big) = \dim \mathbb{C}^d - 2 = 2(d-1),$$

as $\dim N = \dim \mathfrak{n}^* = 1$.

iii. Since $\lambda \in \pi^{-1}(\kappa)$ lies in the interior of $\mu(\mathbb{C}^d)$, it follows that the torus T acts freely, by coordinatewise rotations, on the fiber $\mu^{-1}(\lambda) \subseteq \mathbb{C}^d$. Consequently, the subgroup $H \subseteq T$ acts freely on $\mu^{-1}(\lambda)/N \subseteq \mathbb{C}^d_\kappa$ and thus the action of H on $\mathbb{C}^d_\kappa$ is effective.

□

We are now able to conclude that every Delzant polytope arises as a moment polytope for some symplectic toric manifold.

Proposition 86. *For every Delzant polytope $\Delta \subseteq \mathfrak{t}^*$, there is a toric Hamiltonian manifold (M, ω, T, μ) with $\mu(M) = \Delta$.*

Proof. By Lemma 84, there is a circle subgroup $N \subseteq T$ and an element $\kappa \in \mathfrak{n}^*$ such that $T = H \times N$ and $\Delta = \mu(M)\cap$. By Lemma 85, the partial reduction of (M, ω, T, μ) by N yields a toric Hamiltonian manifold $(M_\kappa, \omega_\kappa, H, \mu_\kappa)$ with $\mu_\kappa(M_\kappa) = \Delta$. □

Exercises

1. Let V be a vector space, let $L \subseteq V$ be a lattice, and let Δ be a Delzant polytope. Show that

   i. $s\Delta$ is a Delzant polytope for all nonzero $s \in \mathbb{R}$,

   ii. $p + \Delta$ is a Delzant polytope for all $p \in V$.

Chapter 9

Connections and Curvature

In this chapter we introduce the language of principal bundles, connections, and curvature. Since we will apply this material to the setting of torus actions on symplectic manifolds, our structure group T will always be a torus. As a consequence, our exposition is substantially more straightforward than the general case, in which the structure group G may be nonabelian.

Our interest in this framework will be made clear in the next chapter, when we study the dependence of the reduction of a Hamiltonian manifold (M, ω, T, μ) on the parameter $\lambda \in \mathfrak{t}^*$. In particular, we will use the fact that, when the action $T \curvearrowright \mu^{-1}(\lambda)$ is free, the projection $\pi : \mu^{-1}(\lambda) \to M_\lambda$ inherits the structure of a T-principal bundle.

Key Points:

1. A T-principal bundle on a smooth manifold M is a fiber bundle $\pi : P \to M$ equipped with a free action $T \curvearrowright P$, such that the orbits of T are the fibers of π.

2. A connection $A \subseteq TP$ on a T-principal bundle $P \to M$ splits the tangent fibers T_uP into a horizontal direction, modeled on T_xM for $x = \pi u$, and a vertical direction, modeled on $\mathfrak{t}$.

3. The curvature $F \in \Omega^2(M, \mathfrak{t})$ of a connection $A \subseteq TP$ measures the variation of A under parallel transport on P. Alternatively, F measures the failure of A to be integrable as a distribution on P.

4. The Chern form $c \in H^2(M, \mathfrak{t})$ of a T-principal bundle $P \to M$ is the cohomology class of the curvature $F \in \Omega^2(M, \mathfrak{t})$ with respect to any connection $A \subseteq TP$. In particular, c does not depend on the choice of the connection $A \subseteq TP$.

Remark. We emphasize that we only consider torus structure groups T. Many of the results in this chapter are not true when the structure group G of $P \to M$ is nonabelian. See Section 9.4 for a discussion of the differences that are encountered in the nonabelian setting.

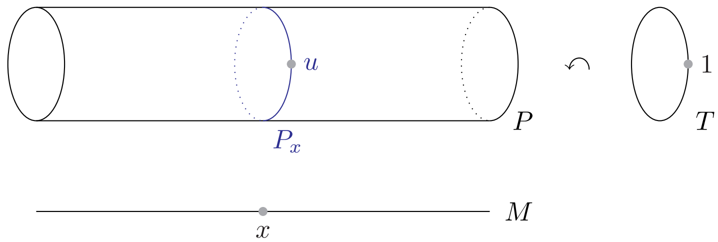
9.1 Principal Bundles and Connections

A *principal bundle* $P \to M$ is a structure which traditionally encodes the global topological properties of a fiber bundle $E \to M$. A *connection* on a principal bundle establishes a distinguished "horizontal" directions at every point $u \in P$, with respect to which we may define a notion of *parallel transport*.

Fix a torus T.

Definition 87. A *T-principal bundle* on a smooth manifold M is a fiber bundle $\pi : P \to M$ equipped with a free action $T \curvearrowright P$, such that the orbits of $T \curvearrowright P$ are precisely the fibers of $\pi : P \to M$.

It follows that the fibers of $P \to M$ are diffeomorphic to T, though there is no canonical identification between P_x and T. Indeed $P \to M$ is naturally a T-fiber bundle with structure group T.



Principal bundles are everywhere. If $E \to M$ is a fiber bundle with typical fiber F and structure group a torus $T \subseteq \operatorname{Diff} E$, then there is an associated principal bundle $P(E) \to M$ with fibers $P(E)_x$ consisting of the admissible identifications of E_x with F,

$$P(E)_x = \{\phi : E_x \xrightarrow{\sim} F\}, \qquad x \in M.$$

The action $T \curvearrowright P(E)$ is given by

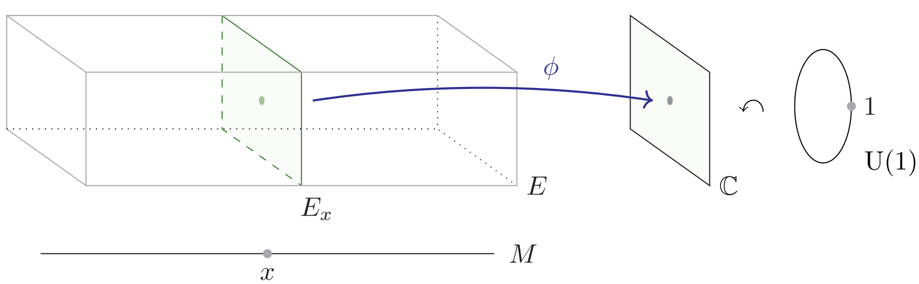
$$t \cdot \phi = t^* \phi : E_x \xrightarrow{\sim} F.$$

Note that the local sections of $P(E) \to M$ correspond to local trivializations of $E \to M$. Informally, we may consider $P(E) \to M$ as the infinitesimal counterpart to the collection of trivializing charts $\{\phi_\alpha : \pi_E^{-1}(U_\alpha) \to U_\alpha \times F\}_\alpha$ of $E \to M$. The original bundle $E \to M$ can be retrieved, up to an isomorphism of fiber bundles, as the associated bundle

$$E \cong P(E) \times_T F.$$

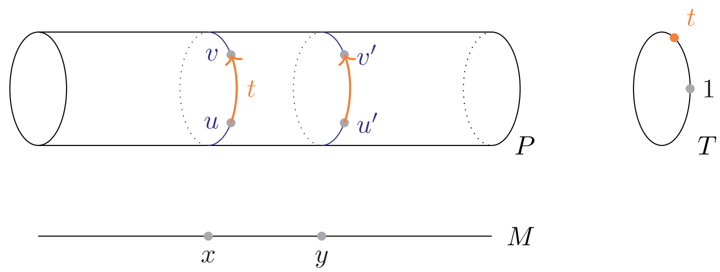
Thus, as the typical fiber F encodes the structure of $E \to M$ over any given $x \in M$, so the principal bundle $P(E) \to M$ encodes the global topological structure of $P \to M$ over all of M.

Example 88. Consider a Hermitian line bundle $E \to M$. In this case, the typical fiber is $F = \mathbb{C}$ equipped with its standard Hermitian structure $\langle \, , \rangle$, and the structure group is $T = \mathrm{U}(1)$. An admissible identification $\phi : E_x \xrightarrow{\sim} \mathbb{C}$ is a linear map which respects the Hermitian structures on E_x and the typical fiber $\mathbb{C}$. The associated principal bundle $P(E) \to M$ consists of all such maps $\phi : E_x \xrightarrow{\sim} \mathbb{C}$ as x ranges over M.



While the associated principal bundle construction $P(E) \to M$ is a common source of principal bundles, we remark that the T-principal bundles that we will encounter in the next chapter will arise in an entirely different manner.

Suppose we wished to identify the points of a fiber P_x with those of a nearby fiber P_y, in a manner that respects the action of T.

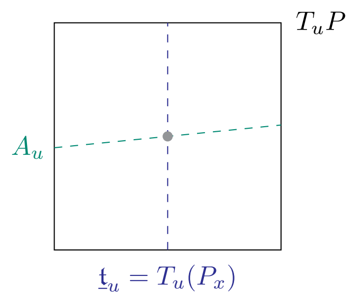


As an alternative to defining a T-equivariant diffeomorphism $a_{x,y} : P_x \to P_y$, we could alternatively take the following infinitesimal approach.

Definition 89. A *connection* on a T-principal bundle $P \to M$ is a distribution $A \subseteq TP$ which is

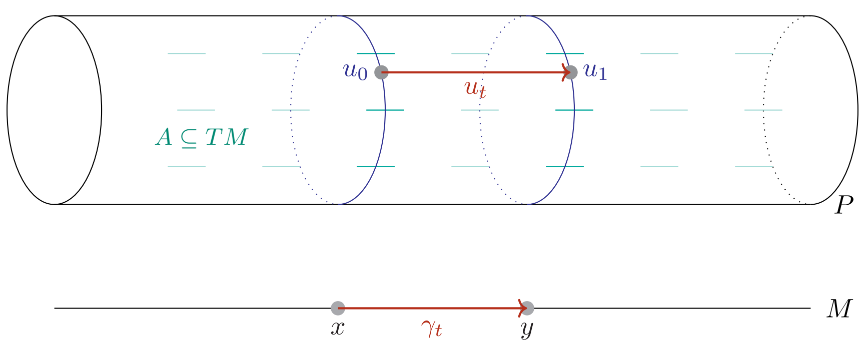
i. T-invariant,
ii. *horizontal*, in the sense that $A_u \oplus \underline{\mathfrak{t}}_u = T_u P$.

The *vertical* directions $\mathfrak{t}_u = T_u(P_x)$ are those that are tangent to the fibers of $P \to M$.



Given a pair of points $x, y \in M$ and a path $\gamma_t : [0,1] \to M$ with $x = \gamma_0$ and $y = \gamma_1$, the *parallel transport* of a element $u_0 \in P_x$ is the endpoint $u_1 \in P_y$, where the path $u_t : [0,1] \to P$ is determined by the condition that

i. u_t covers γ_t, in the sense that $\pi(u_t) = \gamma_t$ for all $t \in [0,1]$,
ii. $\partial_t u$ is always tangent to A.

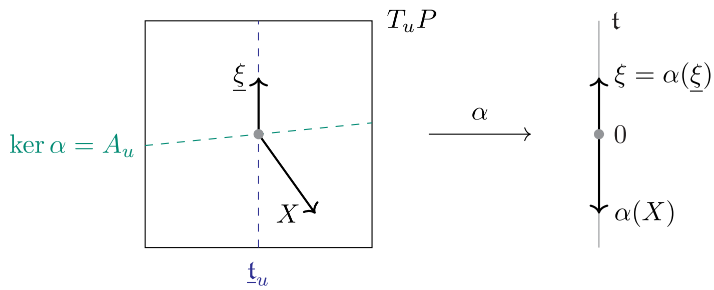


The assignment $u_0 \mapsto u_1$ is T-equivariant since the connection $A \subseteq TP$ is T-invariant. We say that $u_1 \in P_y$ is the *parallel transport* of $x \in P_x$ with respect to the path γ. Note that different paths γ may induce different identifications $P_x \xrightarrow{\sim} P_y$. Informally, the connection $A \subseteq TM$ "connects" the fiber P_x with the fibers $P_{x+\delta x}$ which are infinitesimally nearby. The identification between P_x and P_y, with respect to γ, is achieved by accumulating these infinitesimal identifications along γ.

We frequently express the connection $A \subseteq TP$ as the kernel distribution of a $\mathfrak{t}$-valued 1-form on P.

Definition 90. A *connection* 1*-form* on a T-principal bundle $P \to M$ is a 1-form $\alpha \in \Omega^1(P, \mathfrak{t})$ which is

i. T-invariant,
ii. satisfies $\alpha(\underline{\xi}) = \xi$, for all $\xi \in \mathfrak{t}$, at every point $u \in P$.



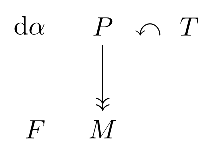
The condition that $\alpha(\underline{\xi}) = \xi$ fixes α along the vertical directions $\underline{\mathfrak{t}} \subseteq TP$. The connection 1-form $\alpha \in \Omega^1(P, \mathfrak{t})$ is completely determined by prescribing a choice of kernel distribution $A = \ker \alpha \subseteq TP$. Indeed, there is a natural bijection

$$\begin{aligned} \{\text{connection 1-forms on } P \to M\} &\xrightarrow{\sim} \{\text{connections on } P \to M\} \\ \alpha \in \Omega^1(M, \mathfrak{t}) &\longmapsto \ker \alpha \subseteq TP \end{aligned}$$

9.2 Curvature

Curvature is the infinitesimal counterpart to the global phenomenon of topological twistedness. We first define curvature in terms of connection 1-forms $\alpha \in \Omega^1(P, \mathfrak{t})$, and then explain it in terms of connections $A \subseteq TP$.

Lemma 91. *If $\pi : P \to M$ is a T-principal bundle, and if $\alpha \in \Omega^1(P, \mathfrak{t})$ is a connection 1-form on P, then there is a unique closed 2-form $F \in \Omega^2(M, \mathfrak{t})$ such that $\pi^*F = \mathrm{d}\alpha$.*



Proof. By the action descent theorem, of Chapter 4, we must show that

i. α is T-invariant,
ii. α is T-horizontal.

It follows that F exists and is unique. The closedness of F follows since π^*F, and since $\pi : P \to M$ is a surjective submersion.

i. This follows from the definition of a connection 1-form.

ii. Fix $\xi \in \mathfrak{t}$. Observe that

- $\mathrm{d}\iota_\xi \alpha = 0$, since $\iota_\xi \alpha \in \mathbb{C}^\infty(P, \mathfrak{t})$ is the function with constant value $\xi \in \mathfrak{t}$,
- $\mathcal{L}_\xi \alpha = 0$, since α is T-equivariant.

Therefore,

$$\iota_\xi \mathrm{d}\alpha = \mathcal{L}_\xi \alpha - \mathrm{d}\iota_\xi \alpha = 0.$$

□

Definition 92. The *curvature* of a connection 1-form $\alpha \in \Omega^2(M, \mathfrak{t})$ on a T-principal bundle $\pi : P \to M$ is the unique closed 2-form $F \in \Omega^2(M, \mathfrak{t})$ satisfying $\mathrm{d}\alpha = \pi^* F$.

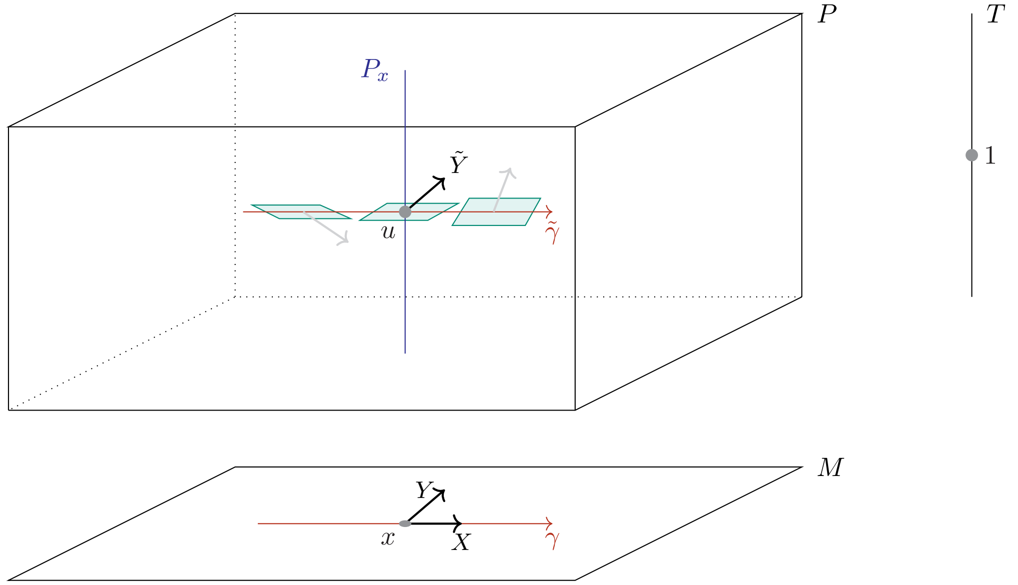
It is instructive to consider the curvature form $F \in \Omega^2(M, \mathfrak{t})$ in terms of the connection $A \subseteq TP$. Let X and Y be local vector fields on M, let $\tilde{X}$ and $\tilde{Y}$ be their unique horizontal lifts to P, and observe that

$$\begin{aligned} \alpha\big([\tilde{X}, \tilde{Y}]\big) &= [\mathcal{L}_{\tilde{X}}, \iota_{\tilde{Y}}]\,\alpha, && \text{since } \alpha\big([\tilde{X}, \tilde{Y}]\big) = \iota_{[\tilde{X},\tilde{Y}]}\,\alpha, \\ &= \mathcal{L}_{\tilde{X}}\iota_{\tilde{Y}}\alpha - \iota_{\tilde{Y}}\mathcal{L}_{\tilde{X}}\alpha \\ &= -\iota_{\tilde{Y}}\iota_{\tilde{X}}\mathrm{d}\alpha, && \text{since } \alpha(\tilde{X}) = \alpha(\tilde{Y}) = 0, \\ &= -F(X, Y). \end{aligned}$$

Therefore,

$$F(X, Y) = -\alpha\big(\mathcal{L}_{\tilde{X}}\tilde{Y}\big).$$

We conclude that $F(X, \cdot\,)$ measures the variation of $A \subseteq TP$ along $\tilde{X}$, in terms of the fiberwise projection $\alpha : TP \to \mathfrak{t}$. In particular, $F_x(X, Y)$ measures this variation in the $\tilde{Y}$ direction.



When the connection $A \subseteq TP$ is integrable as a distribution on P, we have

$$[\tilde{X}, \tilde{Y}] \subseteq A$$

and consequently

$$F(X, Y) = \alpha([\tilde{X}, \tilde{Y}]) = 0$$

for all local vector fields X and Y on M. Thus, we may alternatively interpret the curvature as a measure of the degree to which the connection $A \subseteq TP$ fails to be an integrable distribution.

We are now in a position to clarify the analogy between infinitesimal curvature and global topology. Consider a fiber bundle $E \to M$, modeled on F, and with structure group T. Recall from our discussion above that local sections of $P(E) \to M$ correspond to local trivalizations of $E \to M$. Suppose that $A \subseteq TP$ is a connection on $P \to M$. The subspace $A_u \subseteq TP_u(E)$ is tangent to a local trivialization $\phi : \pi_E^{-1}(U) \to F$ of $E \to M$ on a neighborhood $U \subseteq M$ of $x = \pi u$. Now, the curvature of A measures the infinitesimal incompatibility of the tangent elements to local trivializations $\{A_u\}_{u \in P}$, while the global topology of $E \to M$

is a measure of the inherent non-integrability of a system of local trivializations $\{\phi_U : \pi_E^{-1}(U) \to F\}_{U \subseteq \mathcal{U}}$ for some family of neighborhoods $\mathcal{U}$ covering M.

Before we continue, let us remark that the curvature does not *determine* the global topology of $P \to M$. For example, a principal bundle $P \to M$ may admit a connection $A \subseteq TP$ with curvature $F = 0$, such a connection is said to be *flat*, and still be nontrivial as a fiber bundle on M.

9.3 Characteristic Classes and the Chern–Weil Homomorphism

In this section, we define the *Chern class* of a principal bundle $P \to M$ with structure group a torus T. We then introduce the Chern–Weil homomorphism, which extracts from the curvature form $F \in \Omega^2(M, \mathfrak{t})$ a family of characteristic classes $p([F]) \in H^*(M, \mathbb{R})$ parameterized by multilinear forms $p \in I^*(\mathfrak{t})$.

A *characteristic class* associated to a principal bundle $P \to M$ is a cohomology class on M which describes some aspect of the global topological structure of $P \to M$. As we have taken the perspective that curvature represents, in some sense, the local nontriviality of $P \to M$, it comes as no surprise that characteristic classes may be derived from the curvature. In our current setting, with a torus structure group T, this theory takes a comparatively straightforward form.

Lemma 93. *If $\alpha, \beta \in \Omega^1(P, \mathfrak{t})$ are connection 1-forms on the T-principal bundle $\pi : P \to M$, then there is a unique $\eta \in \Omega^1(M, \mathfrak{t})$ such that $\pi^*\eta = \alpha - \beta$.*

Proof. The difference $\alpha - \beta$ is T-equivariant since α and β are each T-equivariant. Since

$$\iota_\xi(\alpha - \beta) = \xi - \xi = 0, \qquad \xi \in \mathfrak{t},$$

it follows that $\alpha - \beta$ is T-horizontal. Therefore, the action descent theorem implies that η exists and is unique. □

Theorem 94. *If $P \to M$ is a T-principal bundle, then the cohomology class $[F] \in \Omega^2(M, \mathfrak{t})$ does not depend on the choice of connection $A \subseteq TP$.*

Proof. Let F_α and $F_\beta \in \Omega^2(M, \mathfrak{t})$ be the respective curvatures of two connection 1-forms α and $\beta \in \Omega^1(P, \mathfrak{t})$. Lemma 93 implies that the difference

$$F_\alpha - F_\beta = \mathrm{d}(\alpha - \beta)$$

is exact. In particular, the cohomology class $[F_\alpha] = [F_\beta]$ does not depend on α or β. □

Theorem 94 implies that the following definition is well-defined.

Definition 95. The *Chern class* $c \in H^2(M, \mathfrak{t})$ of a T-principal bundle $\pi : P \to M$ is the cohomology class $[F] \in H^2(M, \mathfrak{t})$, where $F \in \Omega^2(M, \mathfrak{t})$ is the curvature of $P \to M$ with respect to any connection $A \subseteq TP$.

Write $I^k(\mathfrak{t})$ for the space of symmetric k-linear forms

$$p : \underbrace{\mathfrak{t} \otimes \cdots \otimes \mathfrak{t}}_{k} \to \mathbb{R}.$$

Given $p \in I^k(\mathfrak{t})$, there is an induced homomorphism of vector bundles

$$\Lambda^{2k} T^*M \otimes \mathfrak{t}^{\otimes k} \xrightarrow{\ p\ } \Lambda^{2k} T^*M,$$

which in turn yields a map

$$\Omega^{2k}(M, \mathfrak{t}^{\otimes k}) \xrightarrow{\ p\ } \Omega^{2k}(M).$$

If $\pi : P \to M$ is a T-principal bundle, if $\alpha \in \Omega^1(P, \mathfrak{t})$ is a connection 1-form on P, and if $F \in \Omega^2(M, \mathfrak{t})$ is the curvature of α, then we denote the image of $F^k \in \Omega^{2k}(M, \mathfrak{t}^{\otimes k})$ under this map by $p(F)$.

Definition 96. The *Chern–Weil homomorphism* with respect to a T-principal bundle $\pi : P \to M$ is the map

$$\begin{aligned} I^*(\mathfrak{t}) &\longrightarrow H^*(M, \mathbb{R}) \\ p &\longmapsto \big[p(F)\big] \end{aligned}$$

We say that $[p(F)] \in H^*(M, \mathbb{R})$ is the *characteristic class* associated to $p \in I^*(\mathfrak{t})$.

Observe that the Chern–Weil homomorphism is well-defined since Theorem 94 ensures that $[F] \in H^2(M, \mathfrak{t})$ does not depend on the choice of connection $A \subseteq TP$.

9.4 The Nonabelian Case

We briefly indicate some of the adjustments that must be made when defining connections, curvature, and characteristic forms when the torus T is replaced by a general compact Lie group G. This material will not appear elsewhere in the course. The interested reader may consult [8, Chapter II] and [9, Chapter XII] for more information.

It is conventional to define a G-principal bundle $P \to M$ to incorporate a *right action* of G on P. This may be motivated, for example, by the observation that the points $u \in P$ are traditionally considered to be dual elements, namely, identifications $u : E_{\pi u} \to F$ associated to an F-fiber bundle $E \to M$, in which setting the symmetries $G \curvearrowright F$ are considered to be primary. When $G = T$ is abelian, every left action is naturally a right action and vice versa. Thus, we ignore this distinction in the exposition above.

Perhaps the most important difference in the nonabelian setting is that a connection 1-form $\alpha \in \Omega^1(P, \mathfrak{g})$ is G-equivariant–as opposed to G-invariant–so that the exterior derivative $\mathrm{d}\alpha \in \Omega^2(P, \mathfrak{g})$ is neither G-invariant nor G-horizontal, and thus does not descend to M.

To accommodate the fact that $\mathrm{d}\alpha$ fails to be

- G-horizontal, we project $\mathrm{d}\alpha_u$ to the horizontal directions $A_u \subseteq T_u P$ at each point $u \in T_u P$,
- G-equivariant, we introduce the *adjoint bundle* $\operatorname{ad} P = P \times_{\mathrm{Ad}} \mathfrak{g} \to M$ which incorporates the action of G.

In this setting, the curvature of a connection A is a 2-form $F \in \Omega^2(M, \operatorname{ad} P)$ taking values in the adjoint bundle $\operatorname{ad} P \to M$. When $G = T$ is abelian, the adjoint action of T on $\mathfrak{t}$ is trivial and there is a natural isomorphism $P \times_{\mathrm{Ad}} \mathfrak{t}$ which the trivial bundle $M \times \mathfrak{t}$. In this case, the curvature takes values in the vector space $\mathfrak{t}$, as we describe above.

As the curvature $F \in \Omega^2(M, \operatorname{ad} P)$ does not take values in $\mathfrak{g}$, the construction of the characteristic class $[p(F)] \in H^*(M, \mathbb{R})$ in the Chern–Weil homomorphism takes place on the total space P. The key property is that, after restricting to G-invariant multilinear forms $p \in I^*(\mathfrak{g})$, the differential form $p(F) \in \Omega^*(P, \mathbb{R})$ is both G-invariant and G-horizontal, and thus descends to M.

Exercises

1. Fix a torus T, a manifold M, and a T-principal bundle $\pi : P \to M$.

   i. Let $U \subseteq M$ be an open set such that $P|_U$ is trivializable. Show that there exists a connection on $P|_U$.

   Hint. Let $P|_U \cong U \times T$ be a trivialization of $P|_U$, so that the action of the structure group T on $U \times T$ is given by left multiplication on the second factor. Using this identification, show that $TU \subseteq T(U \times T)$ defines a connection on $P|_U$.

ii. Show that there exists a connection on $P \to M$.

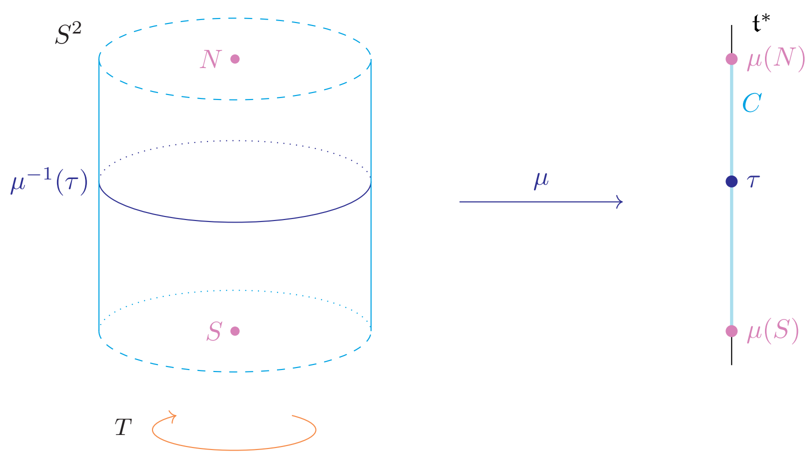
Hint. Let $\mathcal{U} = \{U_i\}_i$ be a locally finite open cover of M which trivializes $P \to M$, and let $\{\psi : U_i \to \mathbb{R}\}_i$ be a partition of unity subordinate to $\mathcal{U}$. Use part i. to establish that, for each $U_i \in \mathcal{U}$, the restriction $P|_{U_i}$ has a connection 1-form $\alpha_i \in \Omega^1(P, \mathfrak{t})$. Now show that $\alpha = \sum_i \psi_i \alpha_i$ is a connection 1-form on $P \to M$.

Chapter 10

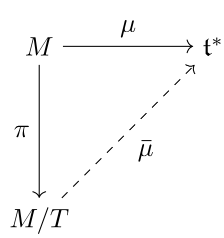
The Duistermaat–Heckman Theorem

The purpose of this chapter is to state and prove the Duistermaat–Heckman theorem. This result describes how the reduced space $(M_\lambda, \omega_\lambda)$ depends on the parameter $\lambda \in \mathfrak{t}^*$.

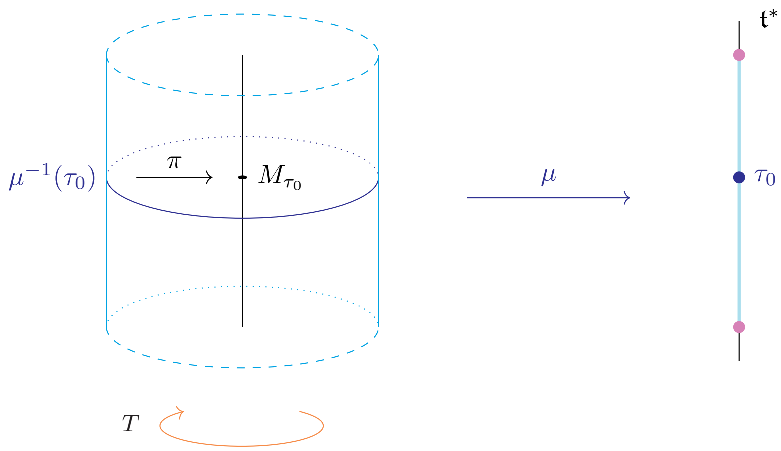
Consider the symplectic sphere. Observe that, over the set $C \subseteq \mathfrak{t}^*$ lying strictly between $\mu(S)$ and $\mu(N)$ the standard moment map defines a fiber bundle with typical fiber S^1.



Furthermore, each fiber $\mu^{-1}(\tau) \subseteq S^2$ inherits an action of the circle T, and that the quotient $\mu^{-1}(\tau)/T$ is the reduced space M_τ. Since the moment map $\mu : M \to \mathfrak{t}^*$ is invariant under the action of $T \curvearrowright M$, there is an induced map $\bar{\mu} : M/T \to \mathfrak{t}^*$.



Over the set $C \subseteq \mathfrak{t}^*$, the map $\bar{\mu} : M/T \to \mathfrak{t}^*$ defines a fiber bundle modeled on M_{τ_0} for any τ_0 between $\mu(S)$ and $\mu(N)$. We will take the perspective that the moment map $\mu : \mu^{-1}(C) \to C$ describes a bundle of T-principal bundles $\mu^{-1}(\tau) \to M_\tau$, modeled on $\mu^{-1}(\tau_0) \to M_{\tau_0}$ for some fixed $\tau_0 \in C$.



As we shall see, it is true in general that there is a canonical isomorphism of the cohomology of the reduced space $H^*(M_\tau)$ with that of the model space $H^*(M_{\tau_0})$. In terms of this identification, the Duistermaat–Heckman theorem establishes a linear relation between the cohomology of the reduced symplectic forms ω_τ, as τ varies over C.

Theorem 97 (Duistermaat–Heckman)**.** *Let (M, ω, T, μ) be a Hamiltonian manifold with T a torus, and let $C \subseteq \mathfrak{t}^*$ be a connected component of the set of regular values of $\mu : M \to \mathfrak{t}^*$. If T acts freely on $\mu^{-1}(C) \subseteq M$, then*

$$[\omega_\tau] = [\omega_{\tau_0}] + \langle \tau - \tau_0, c \rangle,$$

where $c \in H^2(M_\tau, \mathfrak{t})$ is the Chern class of the T-principal bundle $\mu^{-1}(\tau) \to M_\tau$.

We follow the original proof of [6]. The key insight is that a trivialization $\mu^{-1}(C) \xrightarrow{\sim} C \times \mu^{-1}(\tau_0)$ of the moment bundle $\mu^{-1}(C) \to C$, together with the underlying symplectic structure $\omega \in \Omega^2(M)$, induces a distinguished connection on each T-principal bundle $\mu^{-1}(\tau) \to M_\tau$. Defined in terms of ω, these connections unite the symplectic geometry of the reduced spaces M_τ with the curvature of the T-principal bundles $\mu^{-1}(\tau) \to M_\tau$.

Key Points:

1. If (M, ω, T, μ) is a Hamiltonian manifold with T a torus, if $C \subseteq \mathfrak{t}^*$ is a connected set of regular value of $\mu : M \to \mathfrak{t}^*$, and if T acts freely on $\mu^{-1}(C)$, then $\mu^{-1}(\tau) \to M_\tau$ is a T-principal bundle for every $\tau \in C$. We may consider the *moment bundle* $\mu : \mu^{-1}(C) \to C$ as a bundle of T-principal bundles, modeled on $\mu^{-1}(\tau_0) \to M_{\tau_0}$ for any $\tau_0 \in C$.

2. The moment bundle $\mu : \mu^{-1}(C) \to C$ is trivializable. Moreover, a trivialization $\mu^{-1}(C) \xrightarrow{\sim} C \times \mu^{-1}(\tau_0)$ induces a T-principal bundle connection on each fiber $\mu^{-1}(\tau) \to M_\tau$.

3. The *Duistermaat–Heckman theorem* asserts that the cohomology of the reduced form $[\omega_\tau] \in H^2(M_\tau)$ associated to a Hamiltonian manifold (M, ω, T, μ) with T a torus, varies linearly with the parameter $\tau \in \mathfrak{t}^*$.

10.1 The Moment Bundle

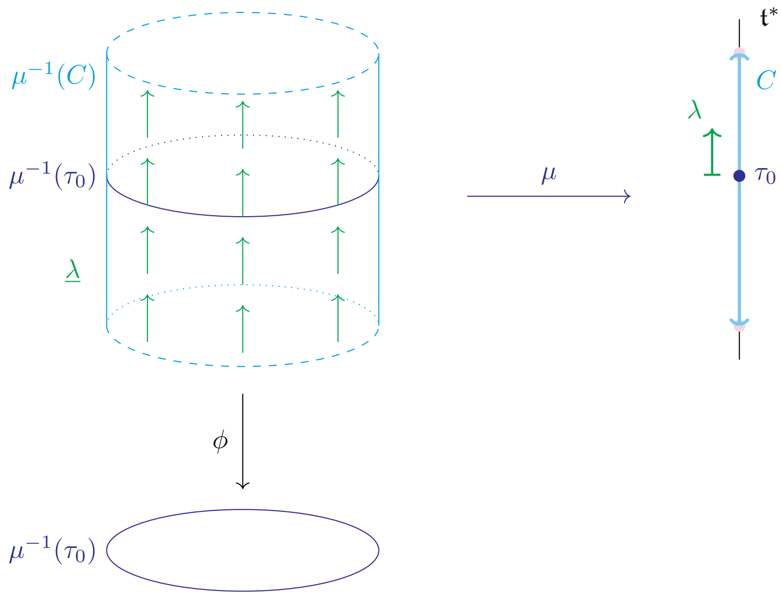
Fix a Hamiltonian manifold (M, ω, T, μ) with T a torus. Let $C \subseteq \mathfrak{t}^*$ be a connected component of the set of regular values of $\mu : M \to \mathfrak{t}^*$, and suppose that T acts freely on $\mu^{-1}(C)$.

Definition 98. The *moment bundle* over C is the restriction of the moment map $\mu : M \to \mathfrak{t}^*$ to C, that is, $\mu : \mu^{-1}(C) \to C$.

Since C is convex, and hence contractible, it follows that the moment bundle is trivializable. Fix a point $\tau_0 \in C$ and a trivializing function $\phi : \mu^{-1}(C) \to \mu^{-1}(\tau_0)$ of $\mu : \mu^{-1}(C) \to C$, so that

$$\mu \times \phi : \mu^{-1}(C) \xrightarrow{\sim} C \times \mu^{-1}(\tau_0).$$

For each $\lambda \in \mathfrak{t}^*$, let $\underline{\lambda} \in \mathfrak{X}\big(\mu^{-1}(C)\big)$ denote the horizontal lift of the constant vector field $\lambda \in \mathfrak{X}(C)$ to $\mu^{-1}(C)$. To keep our notation uniform, and as there is no opportunity for confusion, we will write ι_λ and $\mathcal{L}_\lambda$ for $\iota_{\underline{\lambda}}$ and $\mathcal{L}_{\underline{\lambda}}$, respectively.



In the statement of the Duistermaat–Heckman theorem we add two cohomology classes, $[\omega_\tau]$ and $[\omega_{\tau_0}]$, which lie in distinct spaces $H^2(M_\tau)$ and $H^2(M_{\tau_0})$, respectively. We justify this operation by means of the following lemma.

Lemma 99. *For each $\tau \in C$, there is a natural identification $H^*(M_\tau) \cong H^*(M_{\tau_0})$.*

Proof. Consider the bundle $\mu^{-1}(C)/T \to C$, modeled on the reduced space $M_{\tau_0} = \mu^{-1}(\tau_0)/T$, and fix any two trivializing functions $\psi, \psi' : \mu^{-1}(C)/T \to M_{\tau_0}$. The restriction of ψ and ψ' to the subspace $M_\tau \subseteq \mu^{-1}(C)/T$ yields homotopic diffeomorphisms

$$\psi|_{M_\tau} \simeq \psi'|_{M_\tau} : M_\tau \xrightarrow{\sim} M_{\tau_0},$$

which thus descend to identical isomorphisms of cohomology,

$$\big(\psi|_{M_\tau}\big)_* = \big(\psi'|_{M_\tau}\big)_* : H^*(M_\tau) \xrightarrow{\sim} H^*(M_{\tau_0}).$$

In particular, the isomorphism $H^*(M_\tau) \cong H^*(M_{\tau_0})$ is independent of the choice of trivialization of the bundle $\mu^{-1}(C)/T \to C$. □

In light of Lemma 99, it is meaningful to compare the cohomology classes of the reduced symplectic structures $[\omega_\tau] \in H^2(M_\tau)$ and $[\omega_{\tau_0}] \in H^2(M_{\tau_0})$.

While the trivialization $\mu^{-1}(C) \cong \mu^{-1}(C) \times \mu^{-1}(\tau_0)$ is noncanonical, the induced isomorphism of cohomology $H(M_\tau) \to H(M_{\tau_0})$ is canonical. This is because any two identifications $\phi, \phi' : M$

Since $\mu : \mu^{-1}(C) \to C$ is T-invariant, it follows that $\underline{\lambda} \in \mathfrak{X}\big(\mu^{-1}(C)\big)$ is T-invariant. The vector fields $\underline{\lambda} \in \mathfrak{X}\big(\mu^{-1}(C)\big)$ define a distribution

$$\underline{\mathfrak{t}}^* = \{\underline{\lambda} \,|\, \lambda \in \mathfrak{t}\} \subseteq TM,$$

which is conjugate to the fundamental distribution $\underline{\mathfrak{t}} \subseteq TM$ over $\mu^{-1}(C) \subseteq M$ in the following sense.

Lemma 100. *We have*

$$\omega(\underline{\lambda}, \underline{\xi}) = -\langle \lambda, \xi \rangle,$$

for every $\lambda \in \mathfrak{t}^*$ *and* $\xi \in \mathfrak{t}$.

Proof. Since $\underline{\lambda} \subseteq TM$ lifts the constant vector field $\lambda \in \mathfrak{X}(C)$ to the total space of $\mu : \mu^{-1}(C) \to C$, it follows that $\mu_* \underline{\lambda} = \lambda$. Using the fact that $\omega(X, \underline{\xi}) = -\langle \mu_* X, \xi \rangle$, for all $X \in TM$, we conclude that

$$\omega(\underline{\lambda}, \underline{\xi}) = -\langle \mu_* \underline{\lambda}, \xi \rangle = -\langle \lambda, \xi \rangle.$$

□

Write $\pi : \mu^{-1}(\tau) \to M_\tau$ for the natural projection and $i : \mu^{-1}(\tau) \to M$ for the natural inclusion. Identifying the fibers of $\mu^{-1}(C)/ \to C$ by means of the trivialization $\mu^{-1}(C)/T \cong C \times M_\tau$, we define

$$\partial_\lambda \omega_\tau = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \omega_{\tau + t\lambda} \in \Omega^2(M_\tau),$$

and similarly for $\partial_\lambda i^* \omega$. Observe that, in contrast to the natural isomorphisms of cohomology $H^*(M_\tau) \cong H^*(M_{\tau_0})$, the identifications $M_\tau \cong M_{\tau_0}$ depend on our choice of trivialization $\mu^{-1}(C) \cong C \times \mu^{-1}(\tau_0)$.

The proof of the Duistermaat–Heckman theorem makes use of the following technical lemma.

Lemma 101. *We have*

$$\pi^* \partial_\lambda \omega_\tau = \mathrm{d} i^* \iota_\lambda \omega.$$

Proof. A direct computation yields

$$\begin{aligned}
\pi^* \partial_\lambda \omega_\tau &= \partial_\lambda \pi^* \omega_\tau, && \text{since } \underline{\lambda} \text{ is } T\text{-basic,} \\
&= \partial_\lambda i^* \omega, && \text{by the defining condition } \pi^* \omega_\tau = i^* \omega, \\
&= i^* \mathcal{L}_\lambda \omega, && \text{since } \underline{\lambda} \in \mathfrak{X}\big(\mu^{-1}(C)\big) \text{ lifts } \lambda \in \mathfrak{X}(C), \\
&= i^* \mathrm{d} \iota_\lambda \omega, && \text{using } \mathcal{L}_\lambda = \mathrm{d}\iota_\lambda + \iota_\lambda \mathrm{d} \text{ and } \mathrm{d}\omega = 0, \\
&= \mathrm{d} i^* \iota_\lambda \omega.
\end{aligned}$$

□

10.2 The Chern Class of the Moment Fibers

In this section, we will see that the variation of the cohomology of the reduced form reduced form $\partial_\lambda [\omega_\tau] \in H^2(M_\tau)$ is described by the Chern class $c \in H^2(M_\tau, \mathfrak{t})$ of the T-principal bundle $\mu^{-1}(\tau) \to M_\tau$. The proof of the Duistermaat–Heckman theorem will readily follow from this property.

Our first task is to define a connection 1-form $\alpha \in \Omega^1\big(\mu^{-1}(\tau), \mathfrak{t}\big)$ on the moment fiber $\mu^{-1}(\tau) \to M_\tau$. As we will now show, such a form is induced by our choice of trivialization $\mu^{-1}(C) \cong C \times \mu^{-1}(\tau_0)$.

For each $x \in M$, define the assignment

$$\alpha_x : T_x M \to \mathfrak{t}$$

to be dual to the map

$$\begin{aligned} \mathfrak{t}^* &\to T_x^* M \\ \lambda &\mapsto -\iota_\lambda \omega. \end{aligned}$$

That is, $\alpha \in \Omega^1(M, \mathfrak{t})$ is determined by the condition that

$$\big\langle \lambda, \alpha(X) \big\rangle = -\omega(\underline{\lambda}, X)$$

for all $\lambda \in \mathfrak{t}^*$ and $X \in TM$. Succinctly, $\langle \lambda, \alpha \rangle = -\iota_\lambda \omega$.

Lemma 102. *For each $\tau \in C$, the pullback $i^*\alpha \in \Omega^2\big(\mu^{-1}(\tau), \mathfrak{t}\big)$ is a connection 1-form on $\mu^{-1}(\tau) \to M_\tau$, where $i : \mu^{-1}(\tau) \to M$ is the inclusion.*

Proof. We will show that

i. α is T-invariant,
ii. $\alpha(\underline{\xi}) = \xi$ for all $\xi \in \mathfrak{t}$.

Since the action $T \curvearrowright M$ restricts to $\mu^{-1}(\tau)$, it follows that these properties also hold for $i^*\alpha$.

i. Observe that

$$\begin{aligned} \big\langle \lambda, (t^*\alpha)(X) \big\rangle &= -\omega\big(\underline{\lambda}, t_* X\big) \\ &= -\omega\big(\underline{\lambda}, X\big), \qquad \text{since } \underline{\lambda} \text{ and } \omega \text{ are } T\text{-invariant}, \\ &= \big\langle \lambda, \alpha(X) \big\rangle. \end{aligned}$$

ii. From Lemma 100, we deduce that

$$\big\langle \lambda, \alpha(\underline{\xi}) \big\rangle = -\omega(\underline{\lambda}, \underline{\xi}) = \langle \lambda, \xi \rangle.$$

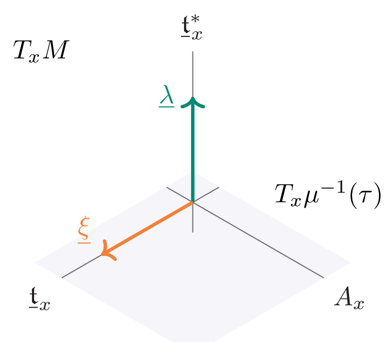
□

For each $\tau \in C$, let $A = \ker \alpha \cap T\mu^{-1}(\tau)$ be the connection on $\mu^{-1}(\tau) \to M_\tau$ corresponding to the restriction of $\alpha \in \Omega^1(M, \mathfrak{t})$ to $\mu^{-1}(\tau)$. Observe that the tangent bundle $T\mu^{-1}(C)$ splits as the sum of three distributions,

$$T\mu^{-1}(C) = \underline{\mathfrak{t}} \oplus \underline{\mathfrak{t}}^* \oplus A,$$

where

- $\underline{\mathfrak{t}}$, which is tangent to the action of T,
- $\underline{\mathfrak{t}}^*$, which is parallel to the reduced space $C \subseteq \mathfrak{t}^*$ of the moment bundle,
- A, which, on each moment fiber $\mu^{-1}(\tau) \to M_\tau$, is parallel to the base M_τ.



Note that in in the example of the sphere (S^2, ω, T, μ), the reduced space M_τ is a point for every $\tau \in C$, and thus the distribution A vanishes on $\mu^{-1}(C)$.

Lemma 103. *For every $\lambda \in \mathfrak{t}^*$, we have*

$$\pi^*\langle \lambda, F \rangle = \mathrm{d}i^*\iota_\lambda \omega,$$

where $F \in \Omega^2(M_\tau, \mathfrak{t})$ is the curvature of $i^*\alpha \in \Omega^1\big(\mu^{-1}(\tau), \mathfrak{t}\big)$, the map $i : \mu^{-1}(\tau) \to M$ is the inclusion, and $\pi : \mu^{-1}(\tau) \to M_\tau$ is the projection.

Proof. This follows as

$$\langle \lambda, \pi^* F \rangle = \langle \lambda, \mathrm{d} i^* \alpha \rangle = \mathrm{d} i^* \langle \lambda, \alpha \rangle = \mathrm{d} i^* \iota_\lambda \omega.$$

□

We are now ready to prove the Duistermaat–Heckman theorem.

Proof of the Duistermaat–Heckman theorem. Lemma 101 and Lemma 103 together yield

$$\pi^* \partial_\lambda \omega_\tau = \mathrm{d} i^* \iota_\lambda \omega = \pi^* \langle \lambda, F \rangle.$$

Since $\pi : \mu^{-1}(\tau) \to M_\tau$ is a surjective submersion, it follows that $\partial_\lambda \omega_\tau = \langle \lambda, F \rangle$. Descending to cohomology on either side yields

$$\partial_\lambda [\omega] = \langle \lambda, c \rangle. \tag{$*$}$$

Since the Chern form $c \in H^2(M_\tau, \mathfrak{t})$ depends only on the T-principal bundle isomorphism class of $\mu^{-1}(\tau) \to M_\tau$, and since Lemma 99 asserts that every fiber of $\mu : \mu^{-1}(C) \to C$ is isomorphic to $\mu^{-1}(\tau_0) \to M_{\tau_0}$, it follows that $c \in H^2(M_\tau, \mathfrak{t}) \cong H^2(M_{\tau_0}, \mathfrak{t})$ is constant as a function of $\tau \in C$. We may thus solve Equation ($*$) with $\lambda = \tau - \tau_0$ to obtain

$$[\omega_\tau] = [\omega_{\tau_0}] + \langle \tau - \tau_0, c \rangle.$$

□

Chapter 11

The Exact Stationary Phase Approximation

Suppose $f \in C^\infty(\mathbb{R})$ is a smooth function with isolated critical points and consider the integral

$$I(t) = \int_{\mathbb{R}} e^{\mathrm{i}tf(x)}\,\mathrm{d}x.$$

The idea behind the *stationary phase approximation* is that, for fixed $t > 0$ very large, the integrand $e^{\mathrm{i}tf(x)}$ oscillates wildly at all points of $\mathbb{R}$ *except* those points $x_0 \in \mathbb{R}$ at which the derivative $f'(x_0)$ vanishes. These are the points at which $e^{\mathrm{i}tf}$ exhibits stationary phase. The effect of this oscillation is to tend to cancel out the contribution to the integral $I(t)$ from all points, other than the critical points $x_0 \in C_f$. In particular, the integral $I(t)$ will depend predominantly on the second-order behavior of f on the critical point set $C_f \subseteq \mathbb{R}$.

Suppose for simplicity that $x_0 \in C_f$ is the unique critical point of f, and consider the Taylor expansion of f at a critical point $x_0 \in C_f$,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + O(x^3).$$

Using the ideas of the stationary phase approximation, we deduce that

$$\begin{aligned}
I(t) &\approx \int_{\mathbb{R}} e^{\mathrm{i}t\left[f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2\right]}\,\mathrm{d}x, && \text{approximating } f \text{ to } 2^{\text{nd}} \text{ order at } x_0,\\
&\approx e^{\mathrm{i}tf(x_0)}\int_{\mathbb{R}} e^{\mathrm{i}t\,\frac{1}{2}f''(x_0)\frac{1}{2}(x-x_0)^2}\,\mathrm{d}x, && \text{since } f'(x_0) = 0,\\
&= e^{\mathrm{i}tf(x_0)}\sqrt{\frac{2\pi\mathrm{i}}{tf''(x_0)}}.
\end{aligned}$$

In the *exact stationary phase approximation* we consider an analogous situation in the setting of Hamiltonian manifolds, in which the approximation is equal to the integrand, that is, the approximation is exact.

Key Points:

1. Given a Lie group action on a smooth manifold $G \curvearrowright M$, there is an associated complex of *G-equivariant differential forms* $\Omega^*_G(M)$ and equivariant exterior derivative $\mathrm{d}_G : \Omega^*_G(M) \to \Omega^{*+1}_G(M)$.

2. If a G-equivariant form $\alpha \in \Omega^*_G(M)$ is G-equivariantly closed, then the *equivariant localization theorem* reduces the integral $\int_M \alpha$ to an integral over the fixed-point set of $G \curvearrowright M$.

3. If (M, ω, G, μ) is a Hamiltonian manifold, then the form $\omega + \mu_\xi \in \Omega^2_G(M)$ is equivariantly closed. The application of the equivariant localization theorem to $e^{\omega+\mu_\xi}$ yields the exact stationary phase approximation.

4. The pushforward of the Liouville measure $e^\omega \in \Omega^*(M)$ by the moment map $\mu : M \to \mathfrak{g}^*$ is called the *Duistermaat–Heckman measure* $\mu_* e^\omega \in \Omega^*(\mathfrak{g}^*)$. The exact stationary phase approximation computes the Fourier transform of this measure.

11.1 Equivariant Differential Forms

In this section we introduce equivariant differential forms. We follow the presentation in [4, Chapter 7].

Definition 104. A *G-equivariant differential form* is a G-equivariant polynomial $\alpha : \mathfrak{g} \to \Omega^*(M)$.

We denote the space of G-equivariant differential forms on M by $\Omega^*_G(M)$. This space possesses a natural grading, given by

$$\deg(p \otimes \alpha) = 2 \deg p + \deg \alpha$$

where $p : \mathfrak{g} \to \mathbb{R}$ is a polynomial, and $\alpha \in \Omega^*$ is an ordinary differential form.

Definition 105. The *equivariant exterior derivative* on $\Omega^*_G(M)$ is the operator

$$\mathrm{d}_G : \Omega^*_G(M) \to \Omega^*_G(M)$$

given by

$$(\mathrm{d}_G\alpha)(\xi) = (\mathrm{d} - \iota_\xi)\, \alpha(\xi)$$

for every $\alpha \in \Omega^*_G(M)$ and $\xi \in \mathfrak{g}$. A G-equivariant differential form $\alpha \in \Omega^*_G(M)$ is said to be *equivariantly closed* if $\mathrm{d}_G\alpha = 0$, and *equivariantly exact* if $\alpha = \mathrm{d}_G\beta$ for some $\beta \in \Omega^*_G(M)$.

The equivariant exterior derivative d_G increases the degree of homogeneous elements $\alpha \in \Omega^*_G(M)$ by one.

Proposition 106. *The pair $(\Omega^*_G(M), \mathrm{d}_G)$ forms a chain complex.*

Proof. For any $\xi \in \mathfrak{g}$, we have

$$\begin{aligned}
(\mathrm{d}_G^2\alpha)(\xi) &= (\mathrm{d} - \iota_\xi)^2\alpha(\xi) \\
&= (\mathrm{d}^2 - \mathrm{d}\iota_\xi - \iota_\xi\mathrm{d} + \iota_\xi^2)\, \alpha(\xi) \\
&= -\mathcal{L}_\xi\, \alpha(\xi), && \text{since } \mathrm{d}^2 = \iota_\xi^2 = 0 \text{ and } \mathcal{L}_\xi = \mathrm{d}\iota_\xi + \iota_\xi\mathrm{d}, \\
&= \alpha\big([\xi, \xi]\big), && \text{by the } G\text{-equivariance of } \alpha, \\
&= 0.
\end{aligned}$$

□

In light of Proposition 106, we define the kth *G-equivariant cohomology module* to be

$$H^k_G = \ker \mathrm{d}_G^{[k]} / \operatorname{im} \mathrm{d}_G^{[k-1]},$$

where $\mathrm{d}_G^{[k]} : \Omega^k_G(M) \to \Omega^{k+1}_G(M)$ denotes the restriction of $\mathrm{d}_G : \Omega^*_G(M) \to \Omega^*_G(M)$ to $\Omega^k_G(M)$.

11.2 The Stationary Phase Approximation

The original version of the stationary phase approximation was proved in [6] as a consequence of the Duistermaat–Heckman theorem. It was later obtained as a consequence of the *equivariant localization theorem* in [2, §7]. Here we present a simplified version of this theorem.

Lemma 107. *If (M, ω, G, μ) is a Hamiltonian manifold, then the equivariant differential form $\omega_G \in \Omega^2_G(M)$ given by*

$$\omega_G(\xi) = \omega + \mu_\xi \in \Omega^*(M), \qquad \xi \in \mathfrak{g},$$

is equivariantly closed.

Proof. For all $\xi \in \mathfrak{g}$, we have

$$\begin{aligned}(\mathrm{d}_G \omega_G)(\xi) &= (\mathrm{d} - \iota_\xi)(\omega + \mu_\xi) \\ &= -\iota_\xi \omega + \mathrm{d}\mu_\xi \\ &= 0,\end{aligned}$$

since $\mathrm{d}\mu_\xi = \iota_\xi \omega$. □

Given a smooth action $G \curvearrowright M$ and a fixed point $x_0 \in M$, define the map

$$\begin{aligned}\mathcal{L}_{x_0}(\xi) : T_{x_0}(M) &\longrightarrow T_{x_0} M \\ X &\longmapsto \mathcal{L}_\xi X.\end{aligned}$$

Here we present the localization theorem for actions with isolated zeros.

Theorem 108 (Equivariant localization, [4] Theorem 7.11)**.** *Let G be a compact group with Lie algebra $\mathfrak{g}$ acting on a compact manifold M, and let α be an equivariantly closed differential form on M. Let $\xi \in \mathfrak{g}$ be such that $\underline{\xi} \in \mathfrak{X}(M)$ has only isolated zeros. Then*

$$\int_M \alpha(\xi) = (-2\pi)^\ell \sum_{x_0 \in M_0(\xi)} \frac{\alpha(\xi)(x_0)}{\det^{1/2}(\mathcal{L}_{x_0}(\xi))},$$

where $\ell = \dim(M)/2$, and by $\alpha(\xi)(x_0)$, we mean the value of the function $\alpha(\xi)_{[0]}$ at the point $x_0 \in M$.

The following version of the exact stationary phase approximation follows immediately.

Theorem 109 (Exact stationary phase approximation)**.** *If (M, ω, G, μ) is a Hamiltonian manifold with M and G compact, and if $\xi \in \mathfrak{g}$ such that $\mu_\xi : M \to \mathfrak{g}^*$ has only isolated zeros, then*

$$\int_M e^{\mathrm{i}\mu_\xi} e^\omega = \sum_{x_0 \in C_{\mu_\xi}} \frac{e^{\mathrm{i}\mu_\xi(x_0)}}{\det^{1/2}(\mathcal{L}_{x_0}(\xi))}.$$

Proof. It follows from Lemma 107 that the element

$$e^{\omega_G} = \sum_{k \geq 0} \frac{\omega_G^k}{k!} \in \Omega^*_G(M)$$

is equivariantly closed. Since the critical point set $C_{\mu_\xi} \subseteq M$ is equal to the fixed point set $M_0(\xi)$, an application of Theorem 108 yields

$$\int_M e^{\mathrm{i}\mu_\xi} e^\omega = \int_M e^{\omega_G}(\xi) = \sum_{x_0 \in C_{\mu_\xi}} \frac{e^{\mathrm{i}\mu_\xi(x_0)}}{\det^{1/2}(L_{x_0}(\xi))}.$$

□

11.3 The Fourier Transform of the Duistermaat–Heckman Measure

In this section, we give an alternative characterization of the exact stationary phase approximation, in terms of the Fourier transform of a distinguished measure on $\mathfrak{g}^*$.

Recall that the canonical measure on a symplectic manifold (M^{2n}, ω) is the *Liouville measure*, which is given as

$$e^{\omega}_{[n]} = \frac{\omega^n}{n!} \in \Omega^{2n}(M).$$

Definition 110. If (M, ω, G, μ) is a Hamiltonian manifold with M compact, then we define the associated *Duistermaat–Heckman measure* measure on $\mathfrak{g}^*$ to be the pushforward by $\mu : M \to \mathfrak{g}^*$ of the Liouville measure $e^{\omega}_{[n]}$ on M.

We will denote the Liouville measure by $e^{\omega} \in \Omega^*(M)$ and the Duistermaat–Heckman measure by $\mu_* e^{\omega} \in \Omega^*(\mathfrak{g}^*)$.

Definition 111. Let V be an n-dimensional vector space. The *Fourier transform* of a measure $m \in \Omega^n(V)$ is the function

$$\widehat{m} : V^* \to \mathbb{C}$$

given by

$$\widehat{m}(\phi) = \int_{x \in V} e^{-\mathrm{i}\langle \phi, x\rangle}\, m, \qquad \phi \in V^*.$$

In particular, if $V = \mathfrak{g}^*$ then we identify $V^* = \mathfrak{g}$ and we define

$$\widehat{m}(\xi) = \int_{\lambda \in \mathfrak{g}^*} e^{-\mathrm{i}\langle \lambda, \xi\rangle}\, m, \qquad \xi \in \mathfrak{g}.$$

Theorem 112 (Fourier transform of the Duistermaat–Heckman measure)**.** *If (M, ω, G, μ) is a Hamiltonian manifold with M and G compact, and if $\xi \in \mathfrak{g}$ such that $\mu_\xi : M \to \mathfrak{g}^*$ has only isolated zeros, then*

$$\widehat{\mu_* e^{\omega}}(\xi) = \sum_{x_0 \in C_{\mu_\xi}} \frac{e^{-\mathrm{i}\mu_\xi(x_0)}}{\det^{1/2}(\mathcal{L}_{x_0}(\xi))}.$$

Proof. From Theorem 109, we derive

$$\begin{aligned}
\widehat{\mu_* e^{\omega}}(-\xi) &= \int_{\lambda \in \mathfrak{g}^*} e^{\mathrm{i}\langle \lambda, \xi\rangle}\, \mu_* e^{\omega} \\
&= \int_{x \in M} e^{\mathrm{i}\langle \mu(x), \xi\rangle}\, e^{\omega} \\
&= \int_M e^{\mathrm{i}\mu_\xi(x)}\, e^{\omega} \\
&= \sum_{x_0 \in C_{\mu_\xi}} \frac{e^{\mathrm{i}\mu_\xi(x_0)}}{\det^{1/2}(\mathcal{L}_{x_0}(\xi))}.
\end{aligned}$$

□

Part III

Geometric Quantization

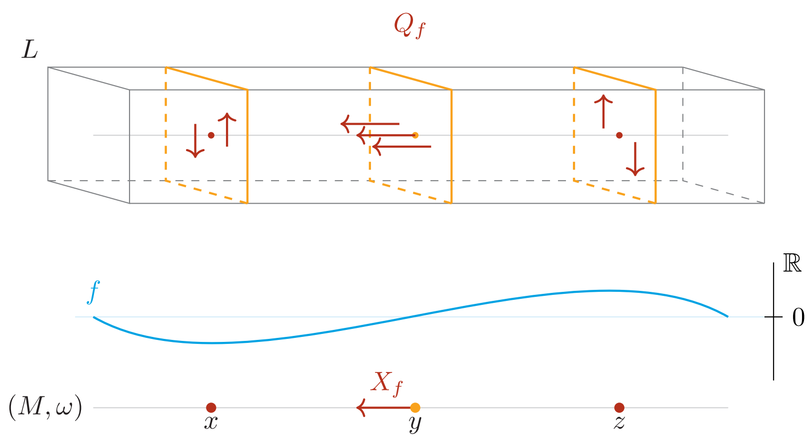
Chapter 12

Prequantization

Quantum physics refers to a family of profoundly mysterious theories about the natural world. *Geometric quantization*, by contrast, comprises an entirely transparent and straightforward collection of mathematical procedures. The basic idea is this:

Lift the symmetries of (M, ω) *to the space of sections of a Hermitian line bundle* $L \to M$.

That is, given a Hamiltonian vector field $X_f \in \mathfrak{X}(M)$ associated to a function $f \in C^\infty(M)$, we would like to extend X_f to an operator Q_f acting on the space of sections $\mathcal{H} = \Gamma(L)$ of a Hermitian line bundle $L \to M$.



More specifically, this describes *geometric prequantization*. The sections of $L \to M$ form a vector space $\mathcal{H}$, and we may consider prequantization as a bridge from the "classical" world of symplectic geometry to the "quantum world" of linear representations.

Classical World	prequantization	Quantum World
symplectic manifolds (M, ω)	$\longrightarrow$	vector spaces $\mathcal{H}$
Hamiltonian symmetries		linear transformations

From this perspective, we might characterize prequantization as the *linearization* of the Hamiltonian symmetries of (M, ω).

There is a complication. The space of sections $\mathcal{H}$ is infinite-dimensional, and we would prefer to work in the finite-dimensional setting. To achieve this, we restrict our attention to certain distinguished subspaces of $\mathcal{H}$. This, however, is the topic of geometric quantization, which we will turn to in the following chapter.

Key Points:

1. Informally, a prequantization of (M, ω) is a lift of the infinitesimal Hamiltonian symmetries $X_f \in \mathfrak{X}(M)$ to infinitesimal symmetries Q_f of the space of sections of a Hermitian line bundle $L \to M$. Alternatively, a prequantization of (M, ω) is a linearization of the Hamiltonian symmetries of (M, ω).

2. A prequantization of (M, ω) is determined by a Hermitian line bundle $L \to M$ with connection ∇, such that $F^\nabla = \mathrm{i}c\,\omega$ for some nonzero constant $c \in \mathbb{R}$.

3. A symplectic manifold (M, ω) can be prequantized if and only if $c[\omega] \in H^2(M, \mathbb{R})$ lies in the image of the inclusion $H^2(M, 2\pi\,\mathbb{Z})$ for some nonzero value $c \in \mathbb{R}$.

4. The quantum observables considered in physics are typically Hermitian, whereas the quantum operators in these notes are skew-Hermitian.

Remark. Though there is substantial interaction between quantum physics and symplectic geometry, we will have little to say about this in these notes. The interested reader may wish to consult [3, 13].

12.1 Lifting the Hamiltonian Symmetries

We begin with a preliminary working definition, which will serve to explain the underlying idea of prequantization. Later on, in Section 12.3, we will replace our working definition with the standard one.

Preliminary Definition. A *prequantization* of a compact symplectic manifold (M, ω) consists of

i. a Hermitian line bundle $L \to M$,

ii. a faithful unitary representation of Lie algebras

$$Q : C^\infty(M) \to \mathfrak{u}(\mathcal{H}),$$

where $\mathcal{H} = \Gamma(L)$ denotes the space of sections of $L \to M$, which lifts the assignment of Hamiltonian vector fields in the sense that

$$Q_f(s \cdot \psi) = (X_f s)\,\psi + s\,Q_f\psi, \tag{$*$}$$

for all $f, s \in C^\infty(M)$ and $\psi \in \mathcal{H}$.

We also impose a technical *first-order* condition on Q: namely, that $(Q_{X_f}\psi)(x) = 0$ for all $\psi \in \mathcal{H}$ whenever f vanishes to first order at $x \in M$.

A few remarks are in order.

i. The Lie algebra structure on $C^\infty(M)$ is defined by the Poisson bracket $\{\,,\}$. The condition that Q is a Lie algebra representation of $C^\infty(M)$ on $\mathcal{H}$ means that $Q : C^\infty(M) \to \mathfrak{u}(\mathcal{H})$ is a homomorphism of Lie algebras.

ii. The Hermitian structure on $\mathcal{H} = \Gamma(L)$ is induced by the Hermitian structure on the fibers of $L \to M$, and by the Liouville measure $\frac{1}{n!}\omega^n$. Explicitly, we define

$$(\psi, \phi)_{\mathcal{H}} = \int_M \langle \psi, \phi \rangle_L \, e^\omega$$

for $\psi, \phi \in \mathcal{H}$. The space of infinitesimal unitary transformations $\mathfrak{u}(\mathcal{H})$ consists of those endomorphisms of $\mathcal{H}$ which generate $(\,,)_{\mathcal{H}}$-preserving transformations of $\mathcal{H}$.

iii. To say that the representation $Q : C^\infty(M) \to \mathfrak{u}(\mathcal{H})$ is *faithful* means that Q_f is the zero operator on $\mathcal{H}$ if and only if f is the zero function on M.

iv. The Hamiltonian lifting property $(*)$ may be understood as follows. The Hamiltonian vector field $X_f \in \mathfrak{X}(M)$ is a derivation on $C^\infty(M)$. That is, $X_f(sh) = (X_f s)h + s(X_f h)$ for all $s, h \in C^\infty(M)$. If we were to extend the domain of X_f to include $\mathcal{H}$, then we must have $X_f(s\psi) = (X_f s)\psi + s(X_f \psi)$ for all $s \in C^\infty(M)$ and $\psi \in \mathcal{H}$. The operator Q_f is precisely such an extension.

v. The first-order condition ensures that the operator Q_f depends on only the first- and zeroth-order behavior of f at any given point of M.

Informally, we consider a function $f \in C^\infty(M)$ as a *classical observable*, and we think of the operator $Q_f \in \mathfrak{u}(\mathcal{H})$ as the associated *quantum observable*. Under this interpretation, a prequantization is an assignment $f \mapsto Q_f$ of a quantum observable Q_f to every classical observable f on (M, ω).

Lemma 113. *If $Q : C^\infty(M) \to \mathfrak{u}(\mathcal{H})$ is a prequantization of a compact connected symplectic manifold (M, ω), then there is a unitary connection ∇ on $L \to M$, and a nonzero constant $c \in \mathbb{R}$, such that*

$$Q_f = \nabla_{X_f} + \mathrm{i}cf,$$

for all $f \in C^\infty(M)$.

Proof. We will show that

i. if $s \in C^\infty(M)$ is a constant function, then $Q_c(\psi) = \mathrm{i}cs\,\psi$ for some $c \in C^\infty(M)$.
ii. The operator

$$\begin{aligned} \nabla : \mathfrak{X}_H(M) \times \mathcal{H} &\longrightarrow \mathcal{H} \\ (X_f, \psi) &\longmapsto Q_f - \mathrm{i}cf, \end{aligned}$$

is well-defined,
iii. $\nabla_{X_f} : \mathcal{H} \to \mathcal{H}$ is a derivation for every $f \in C^\infty(M)$,
iv. ∇ defines a connection on $L \to M$,
v. the function $c \in C^\infty(M)$ is a nonzero constant.

i. For all $s \in C^\infty(M)$ and $\psi \in \mathcal{H}$, we have

$$\begin{aligned} Q_1(s\,\psi) &= (X_1 s)\,\psi + sQ_h\psi, && \text{by the Hamiltonian lifting property } (*), \\ &= s\,Q_1\psi, && \text{since } X_1 = 0. \end{aligned}$$

It follows that the operator $Q_1 : \mathcal{H} \to \mathcal{H}$ is tensorial. Since Q_1 is also unitary, it follows that $Q_1 : \psi \mapsto \mathrm{i}c\psi$ for some function $c \in C^\infty(M)$. The result follows by the linearity of $Q : C^\infty(M) \to \mathfrak{u}(\mathcal{H})$.

ii. If $f, f' \in C^\infty(M)$ satisfy $X_f = X_{f'}$, then $f - f'$ is constant. Part i. implies that

$$Q_{f-f'} = \mathrm{i}c(f - f'),$$

from which we obtain

$$Q_f - \mathrm{i}cf = Q_{f'} - \mathrm{i}cf'.$$

iii. A direct computation yields

$$\begin{aligned} \nabla_{X_f}(s\psi) &= Q_f(s\psi) - \mathrm{i}cf\,s\psi \\ &= (X_f s)\,\psi + sQ_f\psi - s\,\mathrm{i}cf\,\psi, && \text{by the Hamiltonian lifting property } (*) \\ &= (X_f s)\,\psi + s\nabla_{X_f}\psi. \end{aligned}$$

iv. If X_f vanishes at $x \in M$, then $\mathrm{d}f$ vanishes at x. Let $c \in C^\infty(M)$ be the function with constant value $f(x)$. By the first-order condition on Q, it follows that $Q_{f-c}\psi$ vanishes at x for all $\psi \in \mathcal{H}$. Therefore,

$$\nabla_{X_f}\psi = Q_{f-c} - \mathrm{i}c(f-c)$$

vanishes at x for all $\psi \in \mathcal{H}$. Thus, ∇ is tensorial in X_f. In particular, at any point $y \in M$, the map $\psi \mapsto (\nabla_{X_f}\psi)(y)$ depends only on $\psi \in \mathcal{H}$ and on the value of X_f at y. The unitarity of ∇ follows from that of Q.

v. Since $Q : C^\infty(M)$ is a homomorphism of Lie algebras, we have

$$0 = Q_{\{f,1\}}\psi = \left[\nabla_{X_f} + \mathrm{i}cf,\, \mathrm{i}c\right]\psi,$$

and thus

$$0 = \left[\nabla_{X_f},\, c\right]\psi = \left(\nabla_{X_f}c\right)\psi$$

for all $f \in C^\infty(M)$ and $\psi \in \mathcal{H}$. Since M is connected, we deduce that c is a constant function. If $c = 0$, then $Q_c\psi = 0$ for all constant functions $c \in C^\infty(M)$, in violation of the faithfulness of $Q : C^\infty(M) \to \mathfrak{u}(\mathcal{H})$. We therefore conclude that c is nonzero.

□

Informally, Lemma 113 establishes that a quantum operator $Q_f : C^\infty(M) \to \mathfrak{u}(\mathcal{H})$ splits into a

i. *horizontal part*, ∇_{X_f}, defined by means of a unitary connection ∇ on $L \to M$,

ii. *vertical part*, $\mathrm{i}cf$, given in terms of scalar multiplication on the fibers of $L \to M$.

12.2 The Curvature Condition

One of the most fascinating properties of a prequantization $Q : C^\infty(M) \to \mathfrak{u}(\mathcal{H})$ is that the curvature $F^\nabla \in \Omega^2(M, \operatorname{End} L)$ of the associated connection ∇ is always proportional to the symplectic structure $\omega \in \Omega^2(M)$.

Proposition 114. *If $Q : C^\infty(M) \to \mathfrak{u}(\mathcal{H})$ is a prequantization of (M,ω), with $Q_f = \nabla_{X_f} + \mathrm{i}cf$ for each $f \in C^\infty(M)$, then the curvature of ∇ is $F^\nabla = \mathrm{i}c\,\omega$.*

Proof. For $f, h \in C^\infty(M)$, we obtain the following equalities of operators on $\mathcal{H}$,

$$\begin{aligned}
\nabla_{X_{\{f,h\}}} + \mathrm{i}c\,\{f,h\} &= Q_{\{f,h\}} \\
&= [Q_f, Q_h], && \text{since } Q \text{ is a homomorphism of Lie algebras,} \\
&= [\nabla_{X_f} + \mathrm{i}cf, \nabla_{X_h} + \mathrm{i}ch] \\
&= [\nabla_{X_f}, \nabla_{X_h}] + 2\mathrm{i}c\,\{f,h\}, && \text{since } [\nabla_{X_f}, \mathrm{i}c\,h]\psi = \mathrm{i}c\,\{f,h\}\psi \text{ for all } \psi \in \mathcal{H}.
\end{aligned}$$

Rearranging yields

$$-\mathrm{i}c\,\{f,h\} = [\nabla_{X_f}, \nabla_{X_h}] - \nabla_{X_{\{f,h\}}}.$$

Using the fact that $\omega(X_f, X_h) = -\{f,h\}$, on the left-hand side, and that $X_{\{f,h\}} = [X_f, X_h]$ on the right-hand side, we conclude that

$$\mathrm{i}c\,\omega(X_f, X_h) = [\nabla_{X_f}, \nabla_{X_h}] - \nabla_{[X_f,X_h]} = F^\nabla(X_f, X_h).$$

□

12.3 Prequantum Line Bundles

Motivated by the preceding sections, we now introduce the usual definition of a prequantization.

Definition 115. A *prequantization* of a symplectic manifold (M, ω) consists of

i. a Hermitian line bundle $L \to M$,

ii. a unitary connection ∇ on L with curvature $F_\nabla = \mathrm{i}c\,\omega$, for some nonzero constant $c \in \mathbb{R}$.

iii. the assignment

$$\begin{aligned} Q : C^\infty(M) &\longrightarrow \mathcal{D}_1(L) \\ f &\longmapsto Q_f, \end{aligned}$$

where

$$Q_f = \nabla_{X_f} + \mathrm{i}c\,f$$

and where $\mathcal{D}_1(L)$ denotes the space of first-order differential operators on sections of $L \to M$.

In this situation, the pair (L, ∇) called a *prequantum line bundle* on (M, ω), and the operator

$$Q_f = \nabla_{X_f} + \mathrm{i}c\,f$$

is said to be the *quantum operator* associated to $f \in C^\infty(M)$.

Proposition 116. *A compact connected symplectic (M, ω) admits a prequantum line bundle (L, ∇) only if the cohomology class $c\,[\omega] \in H^2(M, \mathbb{R})$ lies in the image of the inclusion $H^2(M, 2\pi\mathbb{Z}) \hookrightarrow H^2(M, \mathbb{R})$.*

Proof. This follows immediately from

i. the condition $F^\nabla = \mathrm{i}c\,\omega$,

ii. the fact that Chern class $[F^\nabla]$ of a Hermitian line bundle lies in the image of the inclusion $H^2(M, 2\pi\mathrm{i}\mathbb{Z}) \hookrightarrow H^2(M, \mathrm{i}\mathbb{R})$.

□

In fact, the converse is also true. That is, a symplectic manifold (M, ω) admits a prequantum line bundle if and only if $c\,[\omega]$ lies in the image of $H^2(M, 2\pi\,\mathbb{Z}) \hookrightarrow H^2(M, \mathbb{R})$.

Remark. In physics, given an observable $f \in C^\infty(M)$, one considers the associated quantum observable

$$\bar{Q}_f = -\mathrm{i}\hbar\,\nabla_{X_f} + f.$$

In terms of our own convention, we have

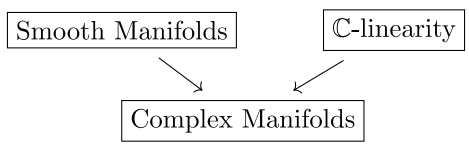
$$\bar{Q}_f = -\mathrm{i}\hbar\,(\nabla_{X_f} + \mathrm{i}\hbar^{-1}\,f) = -\mathrm{i}\hbar\,Q_f.$$

Note that the physicists' operator $\bar{Q}_f$ is not an element of $\mathfrak{u}(\mathcal{H})$. Instead, $\bar{Q}_f$ is a *Hermitian operator* on $\mathcal{H}$, while Q_f is a *skew-Hermitian operator* on $\mathcal{H}$.

Chapter 13

Complex Manifolds

In this chapter we introduce the basic elements complex geometry. Informally, complex geometry is the theory that arises from smooth differential geometry when the properties of complex linearity and complex antilinearity are brought into consideration.



Our interest in complex geometry lies in the formalism of Kähler quantization, to which we will turn in the following chapter. Briefly, the state space of this type of quantization is defined to be the space of holomorphic sections on a Hermitian holomorphic line bundle $L \to M$ over a distinguished type of complex manifold (M, J), called a Kähler manifold. With the exception of Kähler structures, which we introduce in the following chapter, our present task is to explain what all these terms mean.

Key Points:

1. The space of $\mathbb{R}$-linear maps $V = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}^k)$ is a $\mathbb{C}$-vector space. Moreover, V splits as the direct sum $V = V^{1,0} \oplus V^{0,1}$ of the space of $\mathbb{C}$-linear maps $V^{1,0}$ and the space of $\mathbb{C}$-antilinear maps $V^{0,1}$.

2. A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is a smooth function such that the induced map $f_* : T_w\mathbb{C} \to T_{f(w)}\mathbb{C}$ is $\mathbb{C}$-linear at every point $w \in \mathbb{C}$.

3. A complex manifold is a smooth manifold M with a *holomorphic structure*, that is, an atlas of coordinate charts $\phi_i : U_i \to \mathbb{C}^n$ such that the transition functions $\phi_i \circ \phi_j^{-1}$ are biholomorphisms.

4. The *Chern connection* is the unique connection on a Hermitian holomorphic vector bundle $E \to (M, J)$ which is compatible with both the Hermitian and the holomorphic structure of $E \to M$.

Remark. In these notes, we will take a Hermitian structure $\langle\, , \rangle$ to be $\mathbb{C}$-antilinear in the first component and $\mathbb{C}$-linear in the second. In general, our conventions are similar to [12].

13.1 Linear Maps of Complex Vector Spaces

Informally, the global difference between smooth manifolds and complex manifolds, is a consequence of the infinitesimal difference between $\mathbb{R}$-linearizations and $\mathbb{C}$-linearizations. The aim of this section is to describe the

Let $V = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ be the space of $\mathbb{R}$-linear functions from $\mathbb{C}$ to $\mathbb{C}$, and write $V^{1,0}$ (resp. $V^{0,1}$) for the space of $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) maps from $\mathbb{C}$ to $\mathbb{C}$. Denote by V (resp. $V^{1,0}$, $V^{0,1}$) the space of $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear, $\mathbb{C}$-antilinear) maps from $\mathbb{C}$ to $\mathbb{C}$.

V	$\mathbb{R}$-linear
$V^{1,0}$	$\mathbb{C}$-linear
$V^{0,1}$	$\mathbb{C}$-antilinear

Recall that a map of complex vector spaces $A : U \to V$ is $\mathbb{C}$-*antilinear* when

- $A(u + u') = Au + Au'$,
- $A(zu) = \bar{z}Au$.

for all $u, u' \in U$ and $z \in \mathbb{C}$.

Observe that V, $V^{1,0}$, and $V^{0,1}$ are naturally $\mathbb{C}$-vector spaces. Note, for example, that if $A : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-antilinear, then $\mathrm{i}A : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-antilinear as well.

The following key lemma shows that every $\mathbb{R}$-linear function $A : \mathbb{C} \to \mathbb{C}$ splits uniquely as the sum $A = A_- + A_+$ of a $\mathbb{C}$-linear map $A_- : \mathbb{C} \to \mathbb{C}$ and a $\mathbb{C}$-antilinear map $A_+ : \mathbb{C} \to \mathbb{C}$.

Lemma 117. *Put $V = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$. As $\mathbb{C}$-vector spaces, we have $V = V^{1,0} \oplus V^{0,1}$.*

Proof. We will show that

i. $V = V^{1,0} + V^{0,1}$,
ii. $V^{1,0} \cap V^{0,1} = \varnothing$.

i. Fix an $\mathbb{R}$-linear map $A : \mathbb{C} \to \mathbb{C}$, and define $A_\pm : \mathbb{C} \to \mathbb{C}$ by

$$A_\pm(z) = \frac{1}{2}\big[A(z) \pm \mathrm{i}A(\mathrm{i}z)\big], \qquad z \in \mathbb{C}.$$

From

$$\begin{aligned} 2A_-(\mathrm{i}z) &= A(\mathrm{i}z) - \mathrm{i}A(\mathrm{i}^2 z) \\ &= -\mathrm{i}^2 A(\mathrm{i}z) + \mathrm{i}A(z) \\ &= 2\mathrm{i}A_-(z), \end{aligned}$$

and

$$\begin{aligned} 2A_+(\mathrm{i}z) &= A(\mathrm{i}z) + \mathrm{i}A(\mathrm{i}^2 z) \\ &= -\mathrm{i}^2 A(\mathrm{i}z) - \mathrm{i}A(z) \\ &= -2\mathrm{i}A_+(z), \end{aligned}$$

we deduce that A_- is $\mathbb{C}$-linear, and that A_+ is $\mathbb{C}$-antilinear. Since $A = A_- + A_+$, it follows that $V = V^{1,0} + V^{0,1}$.

ii. If $A \in V^{1,0} \cap V^{0,1}$, then

$$\mathrm{i}A(z) = A(\mathrm{i}z) = \bar{\mathrm{i}}A(z),$$

for all $z \in \mathbb{C}$. Since $\bar{\mathrm{i}} = -\mathrm{i}$, it follows that $A(z) = 0$.

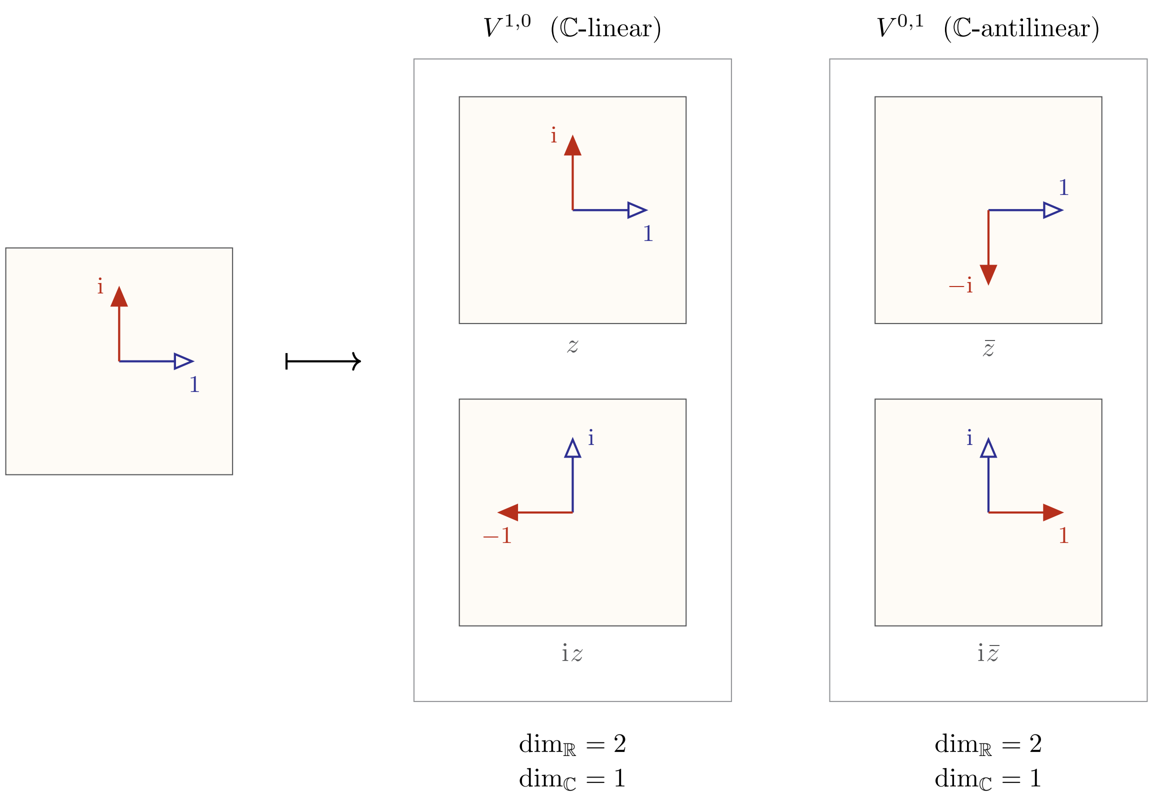
□

Consider the following $\mathbb{R}$-basis of $V = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$.

$$z : \begin{cases} 1 \mapsto 1 \\ \mathrm{i} \mapsto \mathrm{i} \end{cases} \qquad \bar{z} : \begin{cases} 1 \mapsto 1 \\ \mathrm{i} \mapsto -\mathrm{i} \end{cases}$$

$$\mathrm{i}z : \begin{cases} 1 \mapsto \mathrm{i} \\ \mathrm{i} \mapsto -1 \end{cases} \qquad \mathrm{i}\bar{z} : \begin{cases} 1 \mapsto \mathrm{i} \\ \mathrm{i} \mapsto 1 \end{cases}$$

Note that $z : \mathbb{C} \to \mathbb{C}$ is the identity map and $\bar{z} : \mathbb{C} \to \mathbb{C}$ is the identity map composed with complex conjugation. It is conventional to represent by z both the identity map $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ and an arbitrary point in $\mathbb{C}$. This may be compared with the frequent usage of $x, y, t, \ldots$ to denote the identity map $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ and to label points in $\mathbb{R}$.



As $\mathbb{R}$-vector spaces, V is 4-dimensional and $V^{1,0}, V^{0,1}$ are each 2-dimensional. As $\mathbb{C}$-vector spaces, V is 2-dimensional and $V^{1,0}, V^{0,1}$ are each 1-dimensional.

	V	$V^{1,0}$	$V^{0,1}$
$\mathbb{R}$-basis	$\{z, \bar{z}, \mathrm{i}z, \mathrm{i}\bar{z}\}$	$\{z, \mathrm{i}z\}$	$\{\bar{z}, \mathrm{i}\bar{z}\}$
$\mathbb{C}$-basis	$\{z, \bar{z}\}$	$\{z\}$	$\{\bar{z}\}$

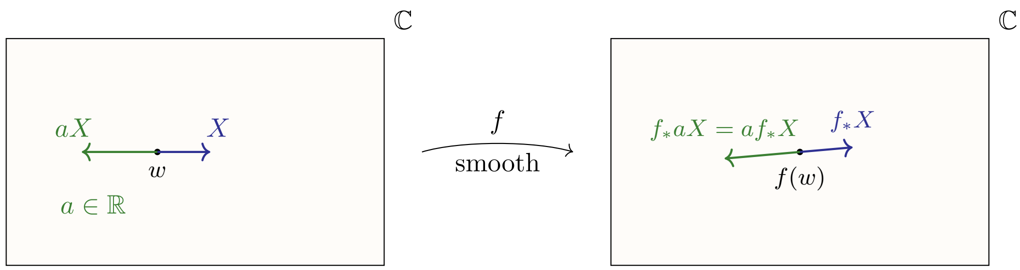
Here we have focused our attention on the underlying complex vector space $\mathbb{C}$. We remark that analogous results hold for finite-dimensional complex vector spaces in the general setting.

13.2 Holomorphic Functions

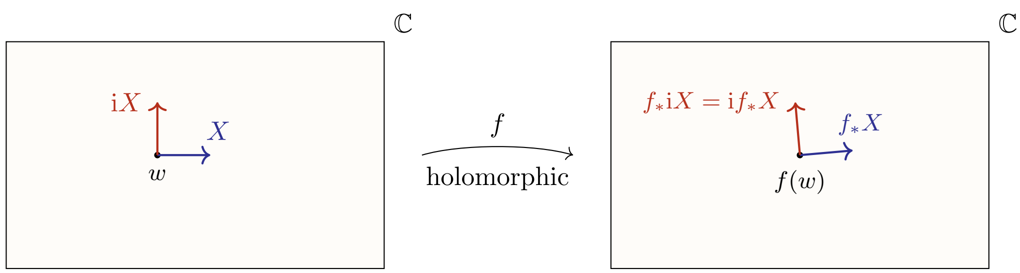
Perhaps the most characteristic feature of complex geometry is the notion of *holomorphicity*. Briefly, a map is holomorphic if its linearization at any point is $\mathbb{C}$-linear. Our present task is to formalize this idea and explore some of its consequences.

Definition 118. A smooth map $f : \mathbb{C} \to \mathbb{C}$ is called *holomorphic* (resp. *antiholomorphic*) if the induced map $f_* : T_w\mathbb{C} \to T_{f(w)}\mathbb{C}$ is $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) at every point $z \in \mathbb{C}$.

Since f is smooth, f_* is $\mathbb{R}$-linear, and thus $f_*(aX) = af_*X$ for every real scalar $a \in \mathbb{R}$.

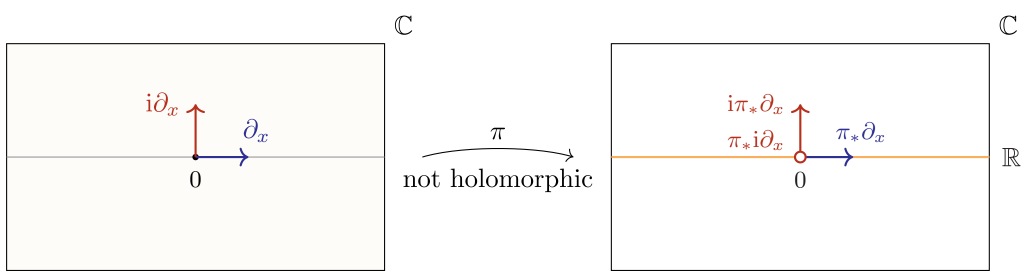


Thus, f is holomorphic precisely when $f_*(\mathrm{i}X) = \mathrm{i}f_*X$ at every $z \in \mathbb{C}$.



To see that this is a nontrivial condition, consider that projection map $\pi : \mathbb{C} \to \mathbb{R}$ given by $(a + b\mathrm{i}) \mapsto a$. At $0 \in \mathbb{C}$, the induced map $\pi_* : T_0\mathbb{C} \to T_0\mathbb{C}$ satisfies

$$\pi_*\mathrm{i}\partial_x = 0 \neq \partial_y = \mathrm{i}\pi_*\partial_x.$$



We now incorporate the properties of $\mathbb{C}$-linearity and $\mathbb{C}$-antilinearity into the construction of cotangent and exterior-algebra bundles. Put $M = \mathbb{C}$.

Definition 119. We define the following $\mathbb{C}$-vector bundles on M.

- The *complexified cotangent bundle* $T^*_{\mathbb{C}}M \to M$ has fiber $\mathrm{Hom}_{\mathbb{R}}(T_wM, \mathbb{C})$ at w consisting of the $\mathbb{R}$-linear maps from T_wM to $\mathbb{C}$.

- The *holomorphic cotangent bundle* $T^{1,0}M \to M$ has fiber $T^{1,0}_w M$ equal to the $\mathbb{C}$-linear maps from $T_w M$ to $\mathbb{C}$.
- The *antiholomorphic cotangent bundle* $T^{0,1}M \to M$ has fiber $T^{0,1}_w M$ consisting of the $\mathbb{C}$-antilinear maps from $T_w M$ to $\mathbb{C}$.
- The *ℓ-holomorphic, k-antiholomorphic exterior algebra bundle* $\Lambda^{k,\ell}M \to M$ is the subbundle of the exterior algebra bundle $\Lambda^{k+\ell}T^*_{\mathbb{C}}M$ with fiber $\Lambda^{k,\ell}_w M$ consisting of elements of the form

$$\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \beta_1 \wedge \ldots \wedge \beta_\ell \in T^*_{\mathbb{C}}M,$$

  where each $\alpha_i : T_w\mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear, and each $\beta_j : T_w M \to \mathbb{C}$ is $\mathbb{C}$-antilinear. We write $\Omega^{k,\ell}(M,\mathbb{C}) = \Gamma(\Lambda^{k,\ell})$ for the space of sections. Elements $\alpha \in \Omega^{k,\ell}(M,\mathbb{C})$ are called *(k,ℓ)-forms* on M.

$T^*_{\mathbb{C}}M$	$\mathbb{R}$-linear
$T^{1,0}M$	$\mathbb{C}$-linear
$T^{0,1}M$	$\mathbb{C}$-antilinear
$\Lambda^{k,\ell}M$	k-linear, ℓ-antilinear

Adapting Lemma 117, we obtain a canonical splitting $T^*_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. This, in turn, induces a splitting on each space of sections,

$$\Omega^m(M,\mathbb{C}) = \bigoplus_{k+\ell=m} \Omega^{k,\ell}(M,\mathbb{C}), \qquad m \geq 0.$$

Suppose $f : \mathbb{C} \to \mathbb{C}$ is a smooth function. Adapting the discussion of the preceding section, we see that the $\mathbb{R}$-linear map $(\mathrm{d}f)_w : T_w\mathbb{C} \to \mathbb{C}$ admits a decomposition in terms of the $\mathbb{C}$-basis $\{\mathrm{d}z, \mathrm{d}\bar{z}\}$.

$\mathbb{C}$	$T_w\mathbb{C}$
V	$(T_{\mathbb{C}}\mathbb{C})_w$
$V^{1,0}$	$T^{1,0}_w\mathbb{C}$
$V^{0,1}$	$T^{0,1}_w\mathbb{C}$
$\{z, \bar{z}\}$	$\{\mathrm{d}z, \mathrm{d}\bar{z}\}$

In particular,

$$(\mathrm{d}f)_w = \mathrm{d}f(\partial_z)\,\mathrm{d}z + \mathrm{d}f(\partial_{\bar{z}})\,\mathrm{d}\bar{z},$$

where $\{\partial_z, \partial_{\bar{z}}\}$ is the dual basis to $\{\mathrm{d}z, \mathrm{d}\bar{z}\}$. We now determine the basis $\{\partial_z, \partial_{\bar{z}}\}$.

Lemma 120. *The dual basis $\{\partial_z, \partial_{\bar{z}}\} \subseteq \mathrm{Hom}_{\mathbb{C}}\big(\mathrm{Hom}_{\mathbb{R}}(T_w\mathbb{C},\mathbb{C}),\mathbb{C}\big)$ to $\{\mathrm{d}z, \mathrm{d}\bar{z}\} \subseteq \mathrm{Hom}_{\mathbb{R}}(T_w\mathbb{C},\mathbb{C})$ is given by*

$$\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$$

and

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y),$$

where $z = x + \mathrm{i}y$.

Proof. While this follows by a direct computation, let us show how to derive it. The defining condition of the dual-basis element ∂_z is that

$$\mathrm{d}z(\partial_z) = 1$$
$$\mathrm{d}\bar{z}(\partial_z) = 0.$$

Writing $z = x + \mathrm{i}y$, we obtain the system of equations

$$(\mathrm{d}x + \mathrm{i}\mathrm{d}y)(\partial_z) = 1$$
$$(\mathrm{d}x - \mathrm{i}\mathrm{d}y)(\partial_z) = 0.$$

Thus,

$$\mathrm{d}x(\partial_z) = \frac{1}{2}, \qquad \mathrm{d}y(\partial_z) = -\frac{\mathrm{i}}{2},$$

so that

$$\mathrm{d}(ax + by)(\partial_z) = \frac{1}{2}(a - \mathrm{i}b) = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)(ax + by)$$

for any $\mathbb{R}$-linear function $ax + by : \mathbb{C} \to \mathbb{C}$ $(a, b \in \mathbb{C})$. We conclude that $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$. The derivation of $\partial_{\bar{z}}$ is similar. □

As a consequence of Lemma 120, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{C}}\big(\mathrm{Hom}_{\mathbb{R}}(T_w\mathbb{C}, \mathbb{C}), \mathbb{C}\big) \cong T_w\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}.$$

Indeed, we generally consider $\{\partial_z, \partial_{\bar{z}}\}$ as a basis of $T_w\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, so that

$$\mathrm{d}f = \underbrace{\frac{\partial f}{\partial z}\,\mathrm{d}z}_{\Omega^{1,0}} + \underbrace{\frac{\partial f}{\partial \bar{z}}\,\mathrm{d}\bar{z}}_{\Omega^{0,1}}.$$

All the above transfers very easily to the setting of smooth maps $f : \mathbb{C}^n \to \mathbb{C}$. The condition of holomorphicity is just as we would expect.

Definition 121. A smooth map $f : \mathbb{C}^n \to \mathbb{C}^k$ is called *holomorphic* (resp. *antiholomorphic*) if the induced map $f_* : T_w\mathbb{C}^n \to T_{f(w)}\mathbb{C}^k$ is $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) at every point $w \in \mathbb{C}$.

We define $\partial f \in \Omega^{1,0}(\mathbb{C}^n, \mathbb{C})$ and $\bar{\partial} f \in \Omega^{0,1}(\mathbb{C}^n, \mathbb{C})$ to be the holomorphic and antiholomorphic parts of $\mathrm{d}f \in \Omega^1(\mathbb{C}^n, \mathbb{C})$, respectively. This suggests the *Dolbeault operators* on $\mathbb{C}^n$,

$$\partial = \pi^{1,0} \circ \mathrm{d} \,:\, C^\infty(\mathbb{C}^n, \mathbb{C}) \to \Omega^{1,0}(\mathbb{C}^n, \mathbb{C})$$

and

$$\bar{\partial} = \pi^{0,1} \circ \mathrm{d} \,:\, C^\infty(\mathbb{C}^n, \mathbb{C}) \to \Omega^{0,1}(\mathbb{C}^n, \mathbb{C}).$$

The space $\mathbb{C}^n$ possesses standard holomorphic coordinates $\{z_1, \dots, z_n\}$, with $z_i = x_i + \mathrm{i}y_i$, and antiholomorphic coordinates $\{\bar{z}_1, \dots, \bar{z}_n\}$, with $z_i = x_i - \mathrm{i}y_i$. When $k = 1$, so that $f : \mathbb{C}^n \to \mathbb{C}$ is a smooth $\mathbb{C}$-valued function, we have

$$\mathrm{d}f = \underbrace{\frac{\partial f}{\partial z_1}\mathrm{d}z_1 + \cdots + \frac{\partial f}{\partial z_n}\mathrm{d}z_n}_{\partial f \,\in\, \Omega^{1,0}} + \underbrace{\frac{\partial f}{\partial \bar{z}_1}\mathrm{d}\bar{z}_1 + \cdots + \frac{\partial f}{\partial \bar{z}_n}\mathrm{d}\bar{z}_n}_{\bar{\partial} f \,\in\, \Omega^{0,1}}.$$

13.3 From Local to Global

We now turn our attention to global setting of complex manifolds. The construction of a complex manifold is analogous to that of a smooth manifold. The only difference is that, given a system of coordinate charts $\phi_i : U_i \to \mathbb{C}^n$, we require the transition functions $\phi_i \circ \phi_j^{-1}$ to be *biholomorphic* on their domains, that is, both holomorphic and possessing a holomorphic inverse.

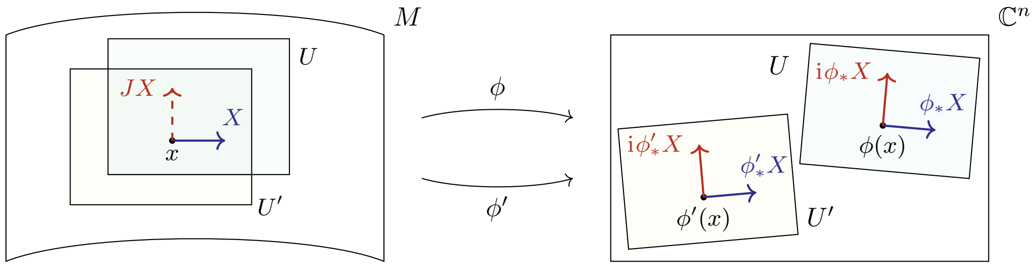
In these notes, we will define a complex manifold to be a smooth manifold M with additional structure.

Definition 122. An n-dimensional *complex manifold* consists of a smooth manifold M with an atlas of charts $\phi_i : U_i \to \mathbb{C}^n$ such that the transition maps

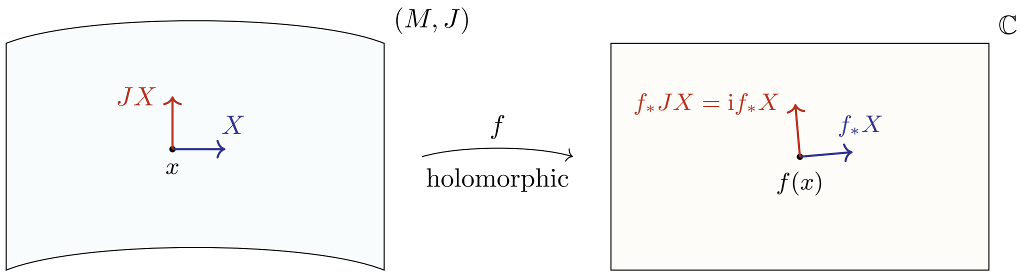
$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

are biholomorphisms. An atlas of this form is called a *holomorphic structure* on M.

A holomorphic structure on M endows the tangent fibers with the structure of a $\mathbb{C}$-vector bundle, in such a way that each coordinate chart $\phi : U \to \mathbb{C}^n$ is a biholomorphism onto its image. We define the *complex structure* $J \in \Gamma(\operatorname{End} TM)$ to be the section of the endomorphism bundle of TM which corresponds to multiplication by i under any coordinate chart $\phi : U \to \mathbb{C}^n$.



Definition 123. A smooth function $f : M \to \mathbb{C}$ is called *holomorphic* if the induced map $f_* : T_xM \to T_{f(x)}\mathbb{C}$ (equivalently, the differential $\mathrm{d}f_x : T_xM \to \mathbb{C}$) is $\mathbb{C}$-linear for all $x \in M$.



More generally, a smooth function $f : (M, J) \to (M', J')$ is called *holomorphic* if $\mathrm{d}\phi_x : T_xM \to T_{\phi(x)}M'$ is $\mathbb{C}$-linear at every $x \in M$.

All the constructions on M which are obtained by introducing a holomorphic structure are straightforward generalizations of the model $\mathbb{C}^n$ setting. Indeed, Definition 119 extends to an arbitrary manifold M without any modification at all. Let us single out one important construction in particular.

Definition 124. We define the *Dolbeault operators* ∂ and $\bar{\partial}$ to be the projection of $\mathrm{d} : C^\infty(M, \mathbb{C}) \to \Omega^1(M, \mathbb{C})$ onto the holomorphic and antiholomorphic subbundles, respectively. That is,

$$\partial = \pi^{1,0} \circ \mathrm{d} \;:\; C^\infty(M, \mathbb{C}) \to \Omega^{1,0}(M, \mathbb{C})$$

and

$$\bar{\partial} = \pi^{0,1} \circ \mathrm{d} \;:\; C^\infty(M, \mathbb{C}) \to \Omega^{0,1}(M, \mathbb{C}).$$

In addition, we write $\mathfrak{X}^{\mathbb{C}}(M) = \Gamma(TM^{\mathbb{C}})$ for the space of sections of the complexified tangent bundle $TM^{\mathbb{C}}$.

13.4 Holomorphic Vector Bundles

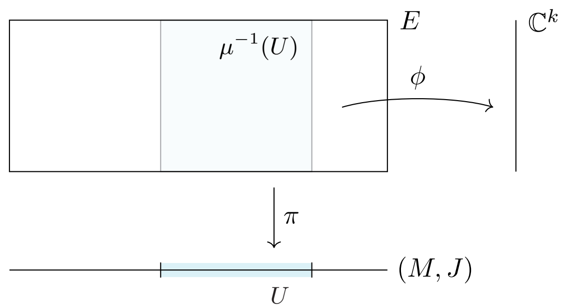
The theory of holomorphic vector bundles is central in the formalism of Kähler quantization, which we will consider in the next chapter. In particular, we will require our prequantum line bundle $L \to (M, \omega)$ to be holomorphic. In this section, we introduce the basic elements of the general theory of holomorphic vector bundles. Our broad aim is to show that every holomorphic vector bundle possesses a distinguished connection ∇ known as the *Chern connection*.

Definition 125. A *holomorphic vector bundle* on (M, J) is a $\mathbb{C}$-vector bundle $E \to M$ such that

i. the total space E is a complex manifold,

ii. the map projection $\pi : E \to M$ is holomorphic,

iii. the local trivializations $\phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ are holomorphic.



Definition 126. The *holomorphic structure* on a holomorphic vector bundle $E \to (M, J)$ is defined to be the map

$$\bar{\partial}_E : \Gamma(E) \to \Omega^1(M, E)$$

given by

$$\bar{\partial}_E \sum_i f_i \sigma_i = \sum_i (\bar{\partial} f_i)\, \sigma_i,$$

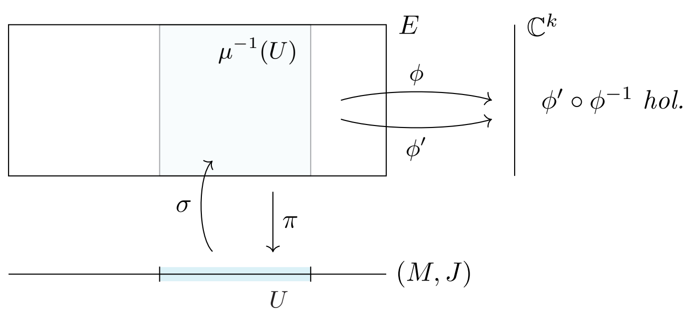
where $f_i \in C^\infty(M)$ and $(\sigma_i)_i$ is a local holomorphic basis of $E \to M$.

Lemma 127. *The operator $\bar{\partial}_E$ is well-defined.*

Proof. By the conditions of a holomorphic vector bundle, the map $\phi' \circ \phi^{-1}$ is holomorphic on its domain. If, additionally, $\phi \circ \sigma$ is holomorphic, then

$$\phi' \circ \sigma = (\phi' \circ \phi^{-1}) \circ (\phi \circ \sigma)$$

is holomorphic as well. $\square$



Remark. Note that we cannot simply define $\bar{\partial}_E$ to be $\pi^{0,1} \circ \mathrm{d}$, since E-valued exterior differentiation $\mathrm{d} : \Gamma(E) \to \Omega^1(M, E)$ is not generally well-defined.

An element $\alpha \in \Omega^1(M, E)$ defines a $\mathbb{R}$-linear map $\alpha_x : T_xM \to E_x$ at every point $x \in M$. As previously, the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathbb{R}}(T_xM, E_x)$ splits as the direct sum of the space of $\mathbb{C}$-linear maps $\mathrm{Hom}_{\mathbb{R}}(T_xM, E_x)^{1,0}$ and the $\mathbb{C}$-antilinear maps $\mathrm{Hom}_{\mathbb{R}}(T_xM, E_x)^{0,1}$. This decomposition, in turn, induces a splitting of the space of E-valued 1-forms,

$$\Omega^1(M, E) = \Omega^{1,0}(M, E) \oplus \Omega^{0,1}(M, E).$$

Definition 128. A *connection* on $E \to M$ is an assignment

$$\begin{aligned} \nabla : \mathfrak{X}^{\mathbb{C}}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (Z, \sigma) &\longmapsto \nabla_Z \sigma, \end{aligned}$$

which is

i. $C^\infty(M, \mathbb{C})$-linear in $Z \in \mathfrak{X}^{\mathbb{C}}(M)$,

ii. $\mathbb{C}$-linear in $\sigma \in \Gamma(E)$,

iii. Leibniz, in the sense that

$$\nabla_Z(f\sigma) = (Zf)\sigma + f\nabla_Z\sigma$$

for all $Z \in \mathfrak{X}^{\mathbb{C}}(M)$, $f \in C^\infty(M, \mathbb{C})$, and $\sigma \in \Gamma(E)$.

We write $\nabla^{1,0}$ and $\nabla^{0,1}$ for the projection of $\nabla : \Gamma(E) \to \Omega^1(M, E)$ onto the holomorphic and antiholomorphic subbundles, respectively. Thus,

$$\nabla^{1,0} = \pi^{1,0} \circ \nabla \,:\, \Gamma(E) \to \Omega^{1,0}(M, E)$$

and

$$\nabla^{1,0} = \pi^{1,0} \circ \nabla \,:\, \Gamma(E) \to \Omega^{0,1}(M, E).$$

Just as the *Levi-Civita connection* is the canonical connection on the tangent bundle of a Riemann manifold, the *Chern connection* is canonical in the setting of Hermitian holomorphic vector bundles.

Theorem 129. *Let $E \to (M, J)$ be a Hermitian holomorphic vector bundle. There is a unique connection ∇ on $E \to M$ which is compatible with the*

i. Hermitian structure*, in the sense that $\langle\, , \rangle_E$ is parallel, so that*

$$\mathrm{d}\langle\sigma, \tau\rangle = \langle\nabla\sigma, \tau\rangle + \langle\sigma, \nabla\tau\rangle$$

for all sections $\sigma, \tau \in \Gamma(E)$,

ii. holomorphic structure*, in the sense that $\nabla^{0,1} = \bar{\partial}_E$.*

The connection ∇ is called the Chern connection *on $E \to (M, J)$.*

Proof. We present the proof of [12, Theorem 4.3]. Suppose ∇ is a connection satisfying conditions i. and ii. We will give an explicit construction of ∇. This will prove both existence and uniqueness.

Let $h : \Gamma(E) \to \Gamma(E^*)$ be the $\mathbb{C}$-antilinear map induced by the Hermitian structure $\langle\, , \rangle_E$. Thus, if $\sigma \in \Gamma(E)$, then $h\sigma \in \Gamma(E^*)$ is defined by

$$h\sigma : \tau \mapsto \langle\sigma, \tau\rangle.$$

Since h is ∇-parallel, we have

$$\nabla_X(h\sigma) = (\nabla_X h)\sigma + h\nabla_X\sigma = h\nabla_X\sigma$$

for all $X \in \mathfrak{X}(M)$. We deduce that

$$\begin{aligned} \nabla_{\mathrm{i}X}(h\sigma) &= \mathrm{i}\nabla_X(h\sigma), && \text{by the } \mathbb{C}\text{-linearity of } \nabla, \\ &= \mathrm{i}h\nabla_X\sigma, && \text{using the previous identity,} \\ &= h\nabla_{-\mathrm{i}X}\sigma, && \text{by the } \mathbb{C}\text{-antilinearity of } h. \end{aligned}$$

It follows that

$$\nabla_Z(h\sigma) = h\nabla_{\bar{Z}}\sigma,$$

for all $Z \in \mathfrak{X}^{\mathbb{C}}(M)$, so that

$$\nabla_{\bar{Z}}\sigma = h^{-1}\nabla_Z(h\sigma).$$

Invoking the fact that $Z \in \mathfrak{X}^{1,0}(M)$ implies that $\bar{Z} \in \mathfrak{X}^{0,1}(M)$, we obtain

$$\nabla^{1,0} = h^{-1}(\nabla^{E^*})^{0,1}h = h^{-1}\bar{\partial}_{E^*}h.$$

Therefore,

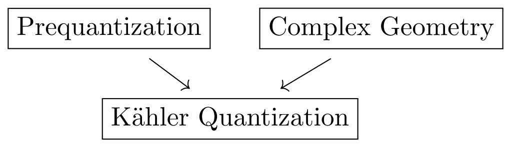
$$\nabla = \nabla^{0,1} + \nabla^{1,0} = \bar{\partial}_E + h^{-1}\bar{\partial}_{E^*}h.$$

$\square$

Chapter 14

Kähler Quantization

Kähler quantization is the natural unification of prequantization and complex geometry.



A Kähler manifold (M, ω, J) is both a symplectic manifold (M, ω) and a complex manifold (M, J) in a compatible manner, which we make precise later on. The Kähler quantization of (M, ω, J) invokes both of these structures in an essential way. Briefly, it consists of a

- **quantum state space** comprising the space of holomorphic sections $\mathcal{H} = \Gamma_{\text{hol}}(L)$ of a prequantum line bundle $(L, \nabla) \to (M, \omega, J)$, where we introduce the condition that ∇ is the Chern connection on $L \to M$,

- **family of quantum operators** $Q_f = \nabla_{X_f} + 2\pi \mathrm{i}$, subject to the constraint that $X_f \in \mathfrak{X}(M)$ preserves the complex structure J, and representing infinitesimal transformations of the Hilbert space $\mathcal{H}$ lifting the classical infinitesimal symmetries X_f of (M, ω, J).

The construction of the quantum state space $\mathcal{H} = \Gamma_{\text{hol}}(L)$ and the quantum operators $Q_f \in \operatorname{End} \mathcal{H}$ fit into a broader notion of quantization: namely, quantization with respect to a polarization. Informally, a polarization is a method for distinguishing one degree of freedom ∂_x on (M, ω) from every conjugate pair $\{\partial_x, \partial_y\}$. This finds its original motivation in physics, where the momentum directions ∂_{p_i} on the momentum phase space $(T^*Q, \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i)$ are traditionally distinguished in each position–momentum pair $\{\partial_{q_i}, \partial_{p_i}\}_i$. The quantum state space is taken to consist of those prequantum states which do not vary in the distinguished, polarized directions.

From this perspective, Kähler quantization distinguishes the antiholomorphic directions $\frac{\partial}{\partial \bar{z}_i}$ from each conjugate pair $\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\}_i$. With respect to the Chern connection ∇, the sections of $L \to M$ which are covariantly constant in the antiholomorphic directions are precisely the holomorphic sections of $L \to M$.

Key Points:

1. The framework of *polarizations* is motivated by physics, where it represents a formal distinction made between position and momentum degrees of freedom in the classical momentum phase space T^*Q.

2. A section σ of a prequantum line bundle $(L, \nabla) \to (M, \omega)$ is said to be *polarized* with respect to a polarization $P \subseteq TM^{\mathbb{C}}$ if σ is covariantly constant along P, that is, $\nabla_P \sigma = 0$. The *quantization* of (M, ω) with respect to P consists of a state space of P-polarized sections $\Gamma_P(L)$, and a family quantum operators $Q_f : \Gamma_P(L) \to \Gamma_P(L)$, for admissible $f \in C^\infty(M)$.

3. A *Kähler manifold* (M, ω, J) possesses mutually compatible symplectic, complex, and Riemannian structures.

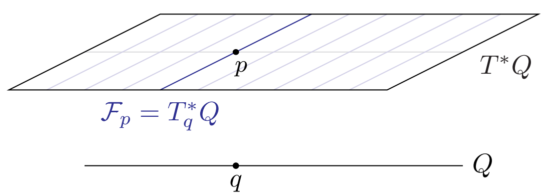
4. The *Kähler quantization* of (M, ω, J) consists of a quantum state space of holomorphic sections of a prequantum line bundle $(L, \nabla) \to M$, and a family of quantum operators $Q_f = \nabla_{X_f} + 2\pi \mathrm{i}$ such that X_f preserves the complex structure J.

Remark. All distributions are assumed to be constant-rank. Note that we frequently extend the symplectic form $\omega \in \Omega^2(M)$ and complex structure $J \in \Gamma(\operatorname{End} TM)$ to the complexified tangent bundle $TM^{\mathbb{C}}$ by $\mathbb{C}$-linearity. Also note that our definition of Kähler quantization may be applied to a broader class of complex symplectic manifolds (M, ω, J): namely, those for which $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in T_x M$, but for which it is not necessarily the case that $g(X, Y) = \omega(X, JY)$ is positive-definite.

14.1 Polarizations

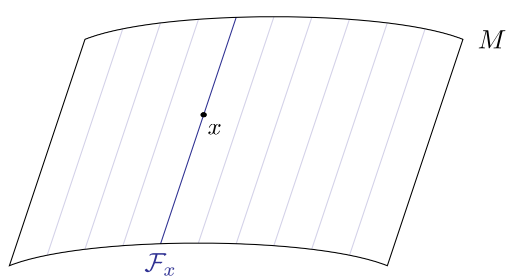
In the physical setting, a meaningful distinction is made between position and momentum. One consequence of the *Heisenberg uncertainty principle* is that a quantum state may be treated to be a function of position, or as a function of momentum, but not both simultaneously. Thus, if the position of a quantum state is precisely determined, then its momentum is entirely unknown.

If a quantum state is treated as a function of position, then we may model this state as a section of a Hermitian line bundle over a classical configuration manifold Q. Note that Q is the leaf space of the Lagrangian foliation $\mathcal{F}$ with leaves $F_p = T_p^*Q$ the cotangent fibers $T^*Q \to Q$.



In fact, we may take quantum states to be defined on the leaf space of *any* Lagrangian foliation of the momentum phase space T^*Q. This motivates the following definition.

Definition 130. A *real polarization* of (M, ω) is a foliation $\mathcal{F}$ of M by Lagrangian submanifolds $\mathcal{F}_x \subseteq M$ $(x \in M)$.



As an intermediate step in the generalization of Definition 130 to the setting of complex manifolds, observe that the tangent bundle $T\mathcal{F} \subseteq TM$ is an integrable Lagrangian distribution on M.

Definition 131. A *complex polarization* of (M, ω) is an integrable Lagrangian distribution of $TM^{\mathbb{C}}$.

Recall that a distribution $P \subseteq TM^{\mathbb{C}}$ is *integrable* if $[P, P] \subseteq P$, and *Lagrangian* if $P_x^{\omega} = P_x$ for every point $x \in M$. We will write $\mathfrak{X}_P(M)$ for the space of sections of $P \subseteq TM^{\mathbb{C}}$. Note that we implicitly extend the symplectic structure $\omega_x : T_x M \times T_x M \to \mathbb{R}$ to a bilinear form $\omega_x : T_x M^{\mathbb{C}} \times T_x M^{\mathbb{C}} \to \mathbb{C}$ by $\mathbb{C}$-linearity.

Definition 132. Let $(L,\nabla) \to (M,\omega)$ be a prequantum line bundle, and let $P \subseteq TM^{\mathbb{C}}$ be a complex polarization of (M,ω). A section $\sigma \in \Gamma(L)$ is said to be *polarized* with respect to P if $\nabla_Z \sigma = 0$ for all $Z \in P$.

That is, σ is polarized with respect to P if σ is covariantly constant along P. Informally, we might consider σ as a function of the leaf space M/P. We will denote the space of P-polarized sections by $\Gamma_P(L) \subseteq \Gamma(L)$.

There are two distinguished classes of complex polarizations:

- $P = \bar{P}$, in which case P is called a *real polarization*,
- $P \cap \bar{P} = \varnothing$, in which case P is said to be a *Kähler polarization*.

Remark. If $P \subseteq TM^{\mathbb{C}}$ is a real complex polarization, then $P = T\mathcal{F}^{\mathbb{C}}$ is the complexified tangent bundle of a real polarization in the sense of Definition 130.

Definition 133. The *quantization* of a symplectic manifold (M,ω) consists of

i. a prequantum line bundle $(L,\nabla) \to M$,

ii. the space of sections $\mathcal{H} = \Gamma_P(L)$ of $L \to M$ which polarized with respect to P,

iii. the operators
$$Q_f = \nabla_{Z_f} + 2\pi \mathrm{i} f$$
on $\Gamma_P(L)$, subject to the condition that X_f preserves P.

Remark. In terms of Definition 115, we are taking $c = 2\pi$. Another common convention is to take $c = 1$.

We now show that the distinguished operators Q_f, with $\mathcal{L}_{X_f} P = 0$, act infinitesimally on the space of polarized sections $\Gamma_P(L)$.

Proposition 134. *If $f \in C^\infty(M)$ preserves P, then $Q_f = \nabla_{X_f} + 2\pi\mathrm{i}$ preserves the space of P-polarized sections $\Gamma_P(L)$.*

Proof. Given any $Z \in \mathfrak{X}_P(M)$, we have

$$\begin{aligned}
\nabla_Z Q_f \sigma &= \nabla_Z \nabla_{X_f} \sigma + 2\pi\mathrm{i} \nabla_Z f \sigma, && \text{as } Q_f = \nabla_{X_f} + 2\pi\mathrm{i} f, \\
&= F^{\nabla}(Z, X_f)\sigma + \nabla_{X_f} \nabla_Z \sigma + \nabla_{[Z,X_f]}\sigma \\
&\qquad + 2\pi\mathrm{i}\,(Zf)\sigma + 2\pi\mathrm{i} f\, \nabla_Z \sigma, && \text{since } F^{\nabla}(Z,X_f) = [\nabla_Z, \nabla_{X_f}] - \nabla_{[Z,X_f]}, \\
&= 2\pi\mathrm{i}\,\omega(Z,X_f)\sigma + 2\pi\mathrm{i}\,\omega(X_f,Z)\sigma, && \text{since } F^{\nabla} = 2\pi\mathrm{i}\,\omega, \\
&= 0.
\end{aligned}$$

It follows that $Q_f\sigma \in \Gamma_P(L)$, and consequently that Q_f generates a 1-parameter transformation of $\Gamma_P(L)$. ☐

14.2 Kähler Manifolds

We now consider a class of manifolds which exhibit a compatible complex and symplectic structure.

Definition 135. A *Kähler structure* on a complex manifold (M,J) is a symplectic form ω such that

i. $\omega(JX, JY) = \omega(X,Y)$,

ii. $g(X,Y) = \omega(X, JY)$ defines a Riemannian metric on M,

for all $X, Y \in T_x M$, $x \in M$. In this case, we say that (M,ω,J) is a *Kähler manifold*.

Remark. Condition i. asserts that ω is preserved by J.

Thus, a Kähler manifold (M, ω, J) is simultaneously a

i. symplectic manifold (M, ω),

ii. complex manifold (M, J),

iii. Riemannian manifold (M, g).

In fact, by the pointwise compatibility condition for Kähler manifolds,

$$g(X, Y) = \omega(X, JY), \qquad X, Y \in T_x M,$$

any two of these structures determines the third.

Lemma 136. *We have* $\overline{T^{0,1}M} = T^{1,0}M$.

Proof. For each $Z \in T^{1,0}M$, we obtain

$$\begin{aligned} J\bar{Z} &= \overline{JZ}, && \text{since } J \text{ is } \mathbb{C}\text{-linear}, \\ &= \overline{-\mathrm{i}Z}, && \text{as } JZ = -\mathrm{i}Z, \\ &= \mathrm{i}\bar{Z}. \end{aligned}$$

It follows that $\bar{X} \in T^{1,0}M$, and consequently that $\overline{T^{0,1}M} \subseteq T^{1,0}M$. This inclusion is an equality since $\bar{\cdot} : TM^{\mathbb{C}} \to TM^{\mathbb{C}}$ is a linear automorphism on fibers and since $\dim T^{0,1}M = \dim T^{1,0}M$. □

Remark. A similar argument shows that $\overline{T^{1,0}M} = T^{0,1}M$.

Proposition 137. *The antiholomorphic tangent bundle* $T^{0,1}M \subseteq TM^{\mathbb{C}}$ *is a Kähler polarization of* (M, ω).

Proof. We will show that

i. $T^{0,1}M$ is integrable,
ii. $T^{0,1}M$ is Lagrangian.
iii. $T^{0,1}M \cap \overline{T^{0,1}M} = \varnothing$.

We prove each statement in turn.

i. For $Z, W \in \Gamma(T^{0,1}M)$, we have

$$\begin{aligned} N_J(Z, W) &= [Z, W] + J[JZ, W] + J[Z, JW] - [JZ, JW] \\ &= [Z, W] + J[-\mathrm{i}Z, W] + J[Z, -\mathrm{i}W] + -[-\mathrm{i}Z, -\mathrm{i}W], && \text{as } JZ = -\mathrm{i}Z, \\ &= 2[Z, W] - 2\mathrm{i}J[Z, W] \end{aligned}$$

Since $N_J = 0$, we rearrange the above equality to obtain

$$J[Z, W] = -\mathrm{i}[Z, W].$$

Consequently, $[Z, W] \in \mathfrak{X}^{0,1}(M)$, and thus $T^{0,1}M$ is integrable.

ii. Given $Z, W \in T^{0,1}_x M$, we have

$$\begin{aligned} \omega(Z, W) &= \omega(JZ, JW), && \text{by Definition 135}, \\ &= \omega(-\mathrm{i}Z, -\mathrm{i}W), && \text{since } JZ = \mathrm{i}W, \\ &= -\omega(Z, W). \end{aligned}$$

It follows that $\omega(Z, W) = 0$, so that $T^{0,1}M$ is isotropic. Since $\dim T^{0,1}M = \frac{1}{2} \dim TM^{\mathbb{C}}$, we conclude that $T^{0,1}M$ is Lagrangian.

iii. By Lemma 136, we must show that $T^{0,1}M \cap T^{1,0}M = \varnothing$. Fix $Z \in TM^{\mathbb{C}}$ with $Z \in T^{1,0}M \cap T^{0,1}M$. From

$$\mathrm{i}Z = JZ = -\mathrm{i}Z$$

we obtain $Z = 0$. Consequently, $T^{1,0}M \cap T^{0,1}M = \varnothing$.

□

Note that proof of Proposition 137 uses only the fact that $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in T_xM$.

Remark. A nearly identical argument shows that the holomorphic tangent bundle $T^{1,0}M \subseteq TM^{\mathbb{C}}$ is also a Kähler polarization of (M, ω).

Lemma 138. *Let $(L, \nabla) \to M$ be a prequantum line bundle on a Kähler manifold (M, ω, J). If ∇ is the Chern connection on $L \to M$, then $\sigma \in \Gamma(L)$ is polarized with respect to $T^{0,1}M \subseteq TM^{\mathbb{C}}$ precisely when σ is holomorphic.*

Proof. We have

$$\begin{aligned} \bar{\partial}\sigma = 0 &\iff \pi^{0,1}\nabla\sigma = 0, && \text{as } \nabla \text{ is the Chern connection,} \\ &\iff \forall Z \in T^{0,1}M : \nabla_Z\sigma = 0. \end{aligned}$$

□

14.3 Quantization

We are now ready to bring everything together. Fix a Kähler manifold (M, ω, J).

Definition 139. The *Kähler quantization* of (M, ω, J) consists of

i. a prequantum line bundle $(L, \nabla) \to M$, such that ∇ the Chern connection,

ii. the space of holomorphic sections $\mathcal{H} = \Gamma_{\mathrm{hol}}(L)$ of $L \to M$,

iii. the operators

$$Q_f = \nabla_{Z_f} + 2\pi\mathrm{i}f$$

on $\Gamma_{\mathrm{hol}}(L)$, subject to the condition that $\mathcal{L}_{X_f}J = 0$.

By Proposition 137 and Lemma 138, the Kähler quantization of (M, ω, J) is a quantization, in the sense of Definition 133, of (M, ω) with respect to the polarization $T^{0,1}M$. In particular, note that $\Gamma_{\mathrm{hol}}(L) = \Gamma_{T^{0,1}M}(L)$.

Our present order of business is to confirm that the quantum operators Q_f do indeed act infinitesimally on the space of quantum states $\Gamma_{\mathrm{hol}}(L)$.

Proposition 140. *If $f \in C^\infty(M)$ satisfies $\mathcal{L}_{X_f}J = 0$, then $Q_f = \nabla_{X_f} + 2\pi\mathrm{i}$ preserves the space of holomorphic sections $\Gamma_{\mathrm{hol}(L)}$.*

Proof. Let $Z \in \mathfrak{X}^{0,1}(M)$ be arbitrary, and observe that

$$J[X_f, Z] = [X_f, JZ] - \underbrace{(\mathcal{L}_{X_f}J)}_{0}(Z) = -\mathrm{i}\,[X_f, Z].$$

It follows that $[X_f, Z] \in T^{0,1}M$, so that X_f preserves $T^{0,1}M$. We conclude by Proposition 134 that Q_f preserves $\Gamma_{\mathrm{hol}}(L) = \Gamma_{T^{0,1}M}(L)$. □

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