Simplicial sets

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Higher Geometry Learning Seminar

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Outline

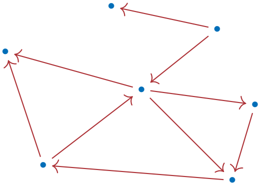
1. Context and motivation

A framework for ∞-categories

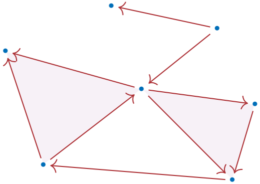
Categories are like *directed graphs*, with

objects $\sim$ vertices

morphisms $\sim$ edges



∞-Categories are like *simplicial complexes*, with

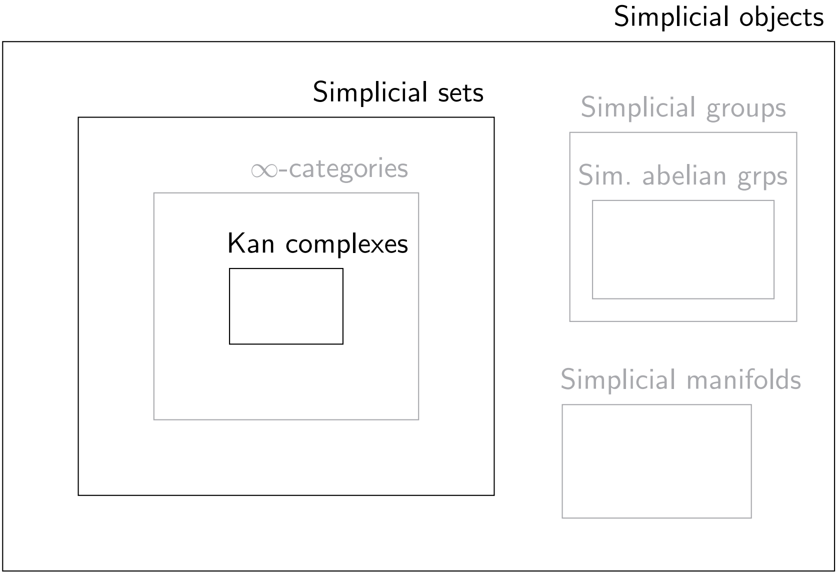


objects $\sim$ vertices

morphisms $\sim$ edges

relations between morphisms $\sim$ higher dimensional faces

Some simplicial constructions



2. The category of simplicial sets

The simplex category

Definition

The simplex category Δ has

objects: sets $[n] = \{0 \leq \cdots \leq n\}$

morphisms: weakly monotone maps $\alpha : [m] \to [n]$

Fix $0 \leq i \leq n$.

- The distinguished inclusions $\delta^i : [n-1] \to [n]$ are

$$\delta^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

- The distinguished projections $\sigma^i : [n+1] \to [n]$ are

$$\sigma^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

Distinguished maps

$$[n+1] = \{1 \leq \cdots \leq i-1 \leq i \leq i+1 \leq i+2 \leq \cdots \leq n+1\}$$

$$\big\downarrow \sigma^i$$

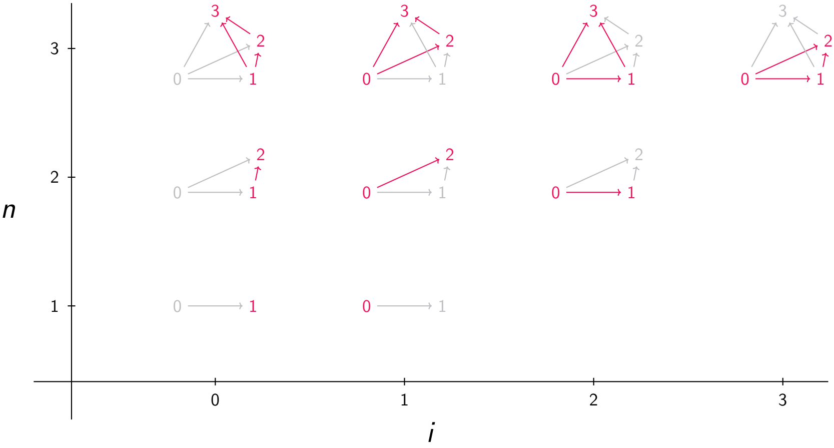
$$[n] = \{1 \leq \cdots \leq i-1 \leq i \leq i+1 \leq \cdots \leq n\}$$

$$\big\uparrow \delta^i$$

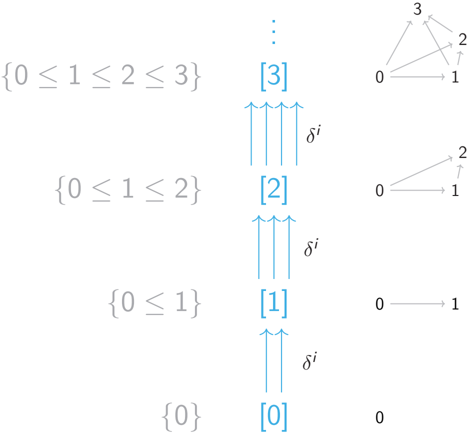
$$[n-1] = \{1 \leq \cdots \leq i-1 \leq i \leq \cdots \leq n-1\}$$

Distinguished inclusions as nondegenerate $(n-1)$-faces

$$\delta^i : [n-1] \to [n]$$



The category Δ and maps δ^i



Simplicial objects

Let $\mathcal{C}$ be a category.

Definition

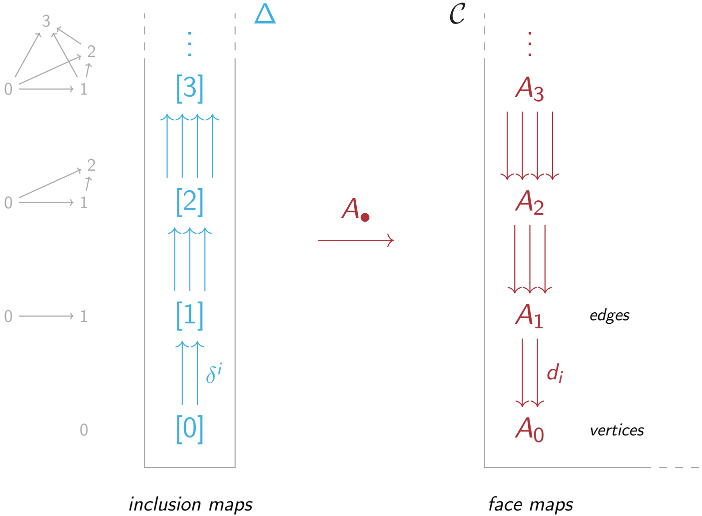
A simplicial object of $\mathcal{C}$ is a functor $A_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$.

- Write $A_n := A_\bullet([n])$ for the n-simplices of $A_\bullet$
- If the objects of $\mathcal{C}$ are denoted $A, B, C, \cdots$, then the simplicial objects are denoted $A_\bullet, B_\bullet, C_\bullet, \cdots$
- In particular, a simplicial set is a presheaf on Δ, i.e. a functor

$$X_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Set} \, .$$

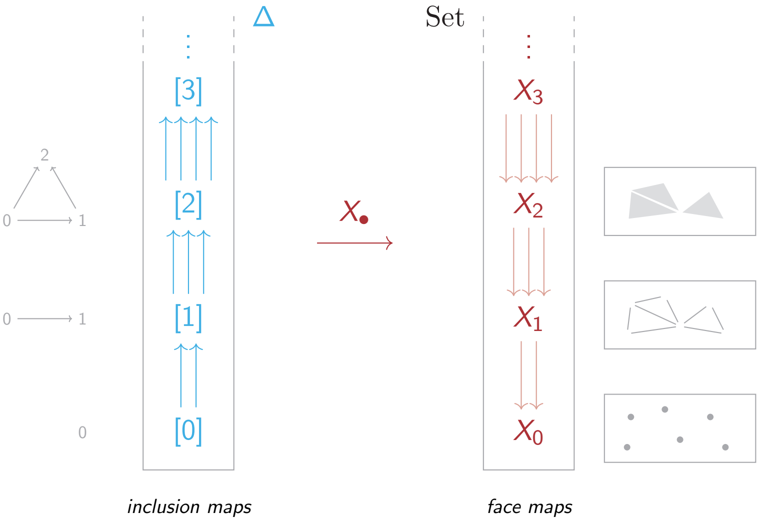
Warning!

A simplicial object of $\mathcal{C}$ is **not** *an object of $\mathcal{C}$.*



$A_\bullet(\delta^i) = d_i$ face maps

$A_\bullet(\sigma^i) = s_i$ degeneracy maps


2
0 1
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0
Δ
⋮
[3]
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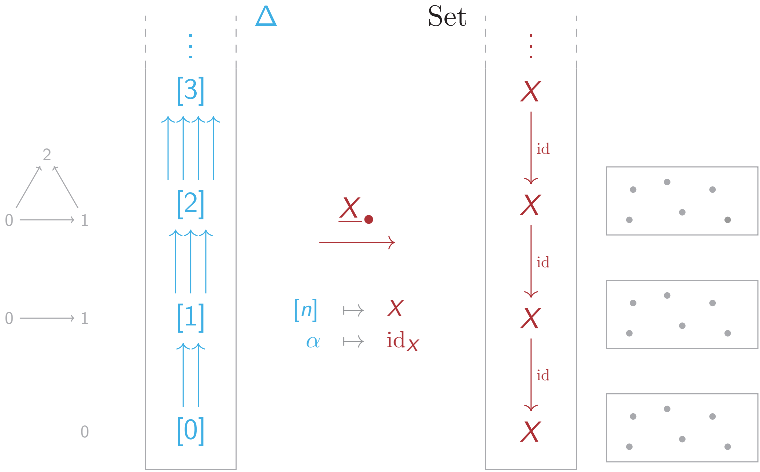
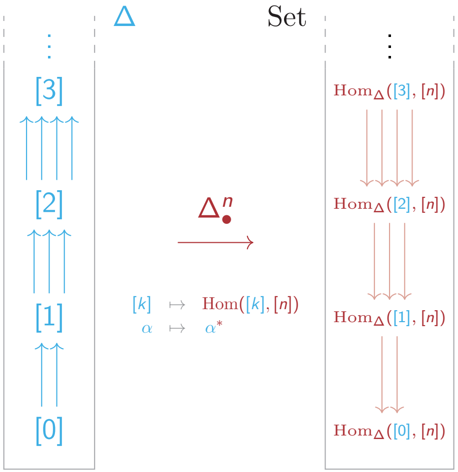
[1]
[0]
inclusion maps
X•
Set
⋮
X3
X2
X1
X0
face maps

Example: constant simplicial set



Example: standard n-simplex



Simplicial morphisms

Definition

A simplicial morphism from

$$A_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C} \qquad \text{to} \qquad B_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$$

is a natural transformation $f_\bullet : A_\bullet \Longrightarrow B_\bullet$.

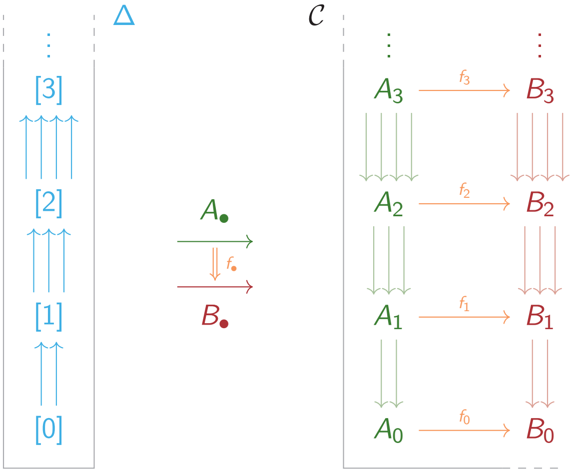
- We arrive at the category of simplicial objects of $\mathcal{C}$,

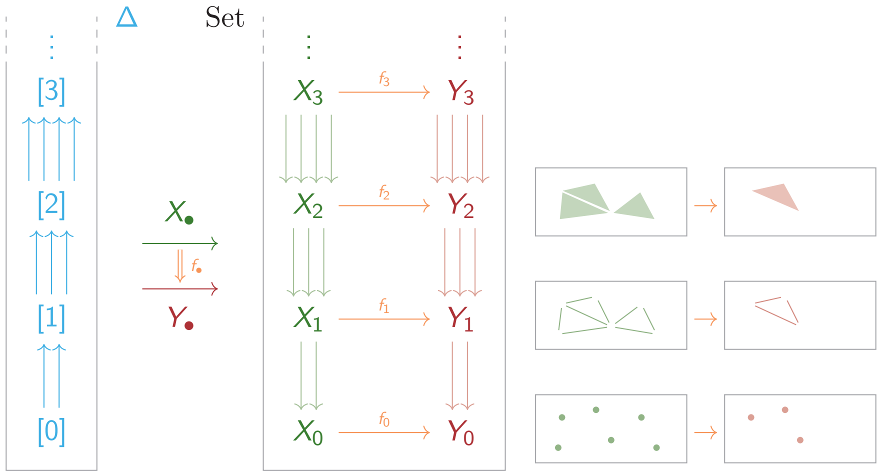
$$\mathcal{C}_\Delta = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$$

also denoted $\mathrm{s}\mathcal{C}$, $\mathrm{S}\mathcal{C}$, ...

$$\mathrm{Hom}_{\mathcal{C}_\Delta}(A_\bullet, B_\bullet) = \mathrm{Nat}(A_\bullet, B_\bullet)\ .$$

- For example, $\mathrm{Set}_\Delta,\ \ \mathrm{Ab}_\Delta,\ \ \mathrm{Mfd}_\Delta$


Δ
[3]
[2]
[1]
[0]
$A_\bullet$
$f_\bullet$
$B_\bullet$
$\mathcal{C}$
A_3
f_3
B_3
A_2
f_2
B_2
A_1
f_1
B_1
A_0
f_0
B_0



Δ
Set
[3]
[2]
[1]
[0]
$X_\bullet$
$f_\bullet$
$Y_\bullet$
X_3
X_2
X_1
X_0
f_3
f_2
f_1
f_0
Y_3
Y_2
Y_1
Y_0

Reminder: Yoneda Lemma

Yoneda Lemma

For any $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and $A \in \mathcal{C}$, there is a canonical bijection

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, A), F) \cong F(A)$$
$$(\mathrm{id}_A \mapsto x) \quad \leftrightarrow \quad x$$

Moreover, this bijection is natural in A and F. (Note: $\mathcal{C}$ must be locally small.)

- The *Yoneda embedding* $y : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is given by

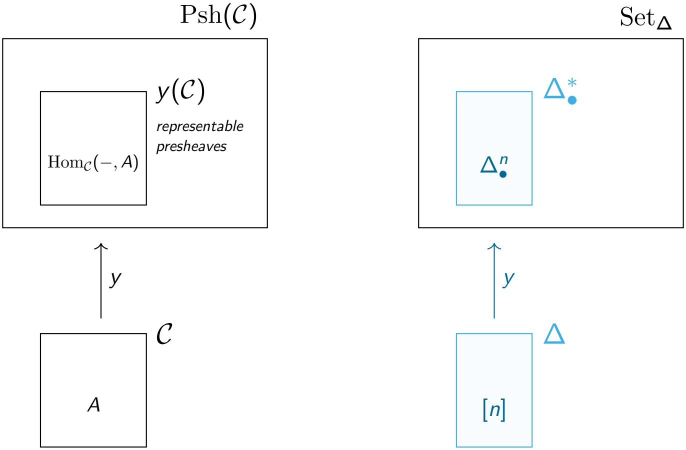
$$y(A) = \mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \to \mathrm{Set}$$

- When $\mathcal{C} = \Delta$,

$$\underset{\mathrm{Nat}(\mathrm{Hom}(-,[n]),B_\bullet)}{\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n_\bullet, B_\bullet)} \cong \underset{B_\bullet([n])}{B_n}$$

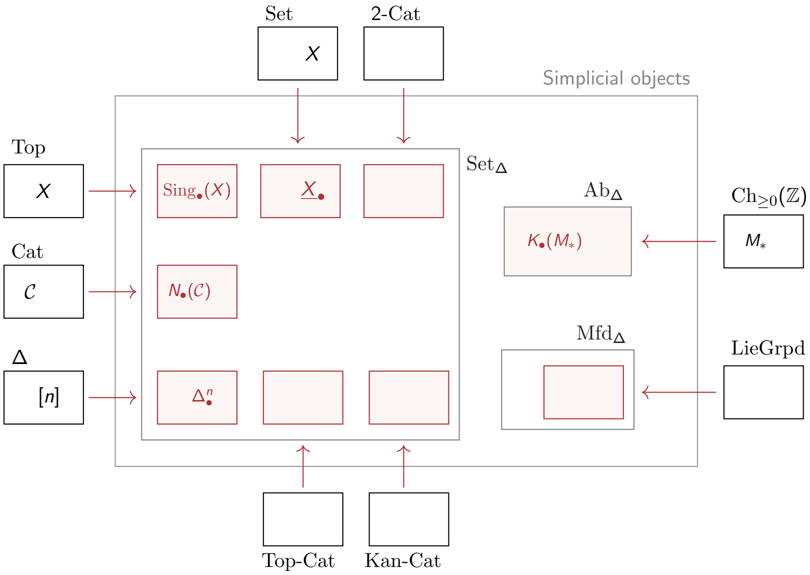
(unique nondegenerate n-face $\mapsto x$) $\leftrightarrow$ x

Yoneda embedding



3. Nerves

Sources of simplicial sets



Idea of the nerve construction

A method for realizing objects $A \in \mathcal{C}$ as simplicial sets $N_\bullet(A) \in \mathrm{Set}_\Delta$.

1. Insert Δ into $\mathcal{C}$, i.e. define a cosimplicial object
$$C^\bullet : \Delta \to \mathcal{C}\,.$$
2. Pull back $y(A) = \mathrm{Hom}_{\mathcal{C}}(-, A)$ along $C^\bullet$, i.e. define
$$N_\bullet(A) = \mathrm{Hom}_{\mathcal{C}}(C^\bullet, A)\,.$$

$N_\bullet =$ "restrict the Yoneda embedding along a cosimplicial object"

Cosimplicial objects

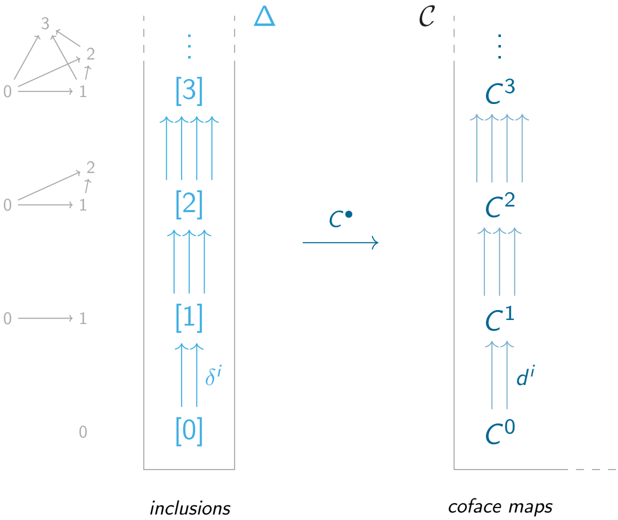
Let $\mathcal{C}$ be category.

Definition

A cosimplicial object of $\mathcal{C}$ is a functor $A^\bullet : \Delta \to \mathcal{C}$.

- Write $A^n := A^\bullet([n])$.
- If the objects of $\mathcal{C}$ are denoted $A, B, C, \cdots$, then the cosimplicial objects are denoted $A^\bullet, B^\bullet, C^\bullet, \cdots$ or, alternatively, as $\Delta^\bullet_{\mathcal{C}}$
- Responds to the question:

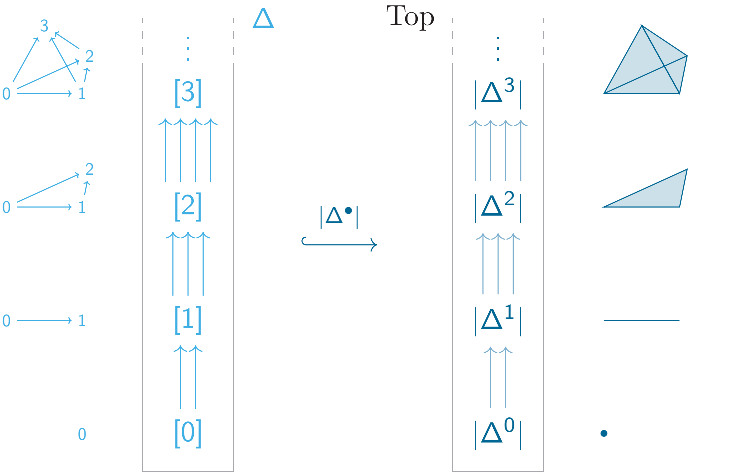
What does Δ "look like" in $\mathcal{C}$?



$A_\bullet(\delta^i) = d^i$ coface maps

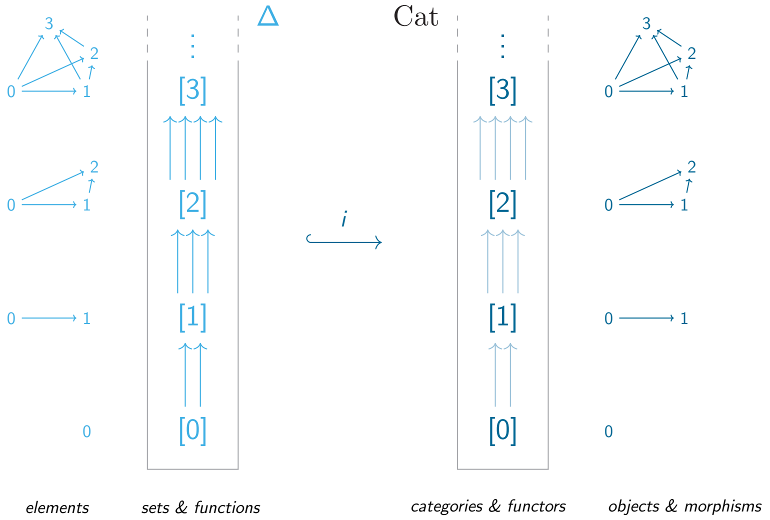
$A_\bullet(\sigma^i) = s^i$ codegeneracy maps

Example: topological simplex functor

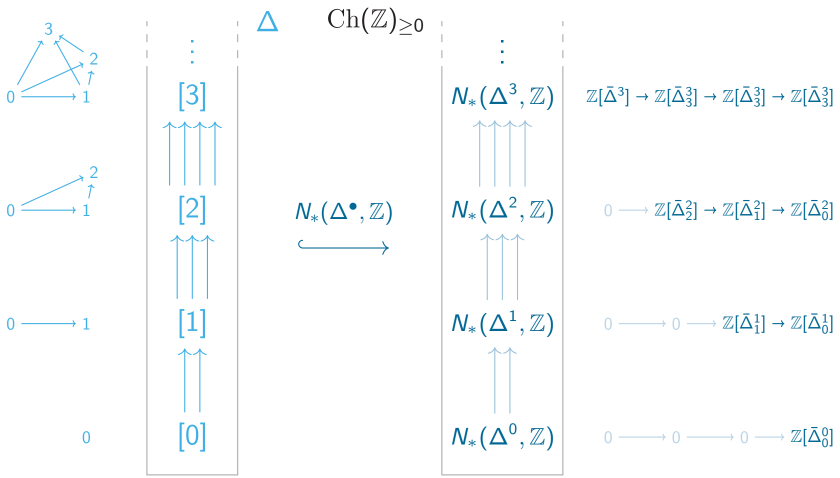


$$|\Delta^n| = \mathrm{Conv}(e_1, \ldots, e_{n+1}) \subseteq \mathbb{R}^{n+1}$$

Example: $\Delta \hookrightarrow \mathrm{Cat}$



Example: normalized Moore complex of Δ^n



$\bar{\Delta}^n$ are the nondegenerate faces of Δ^n

$\partial = \sum_i (-1)^i d_i$

The nerve construction

Fix a

1. cosimplicial object $C^\bullet : \Delta \to \mathcal{C}$,
2. object $A \in \mathcal{C}$.

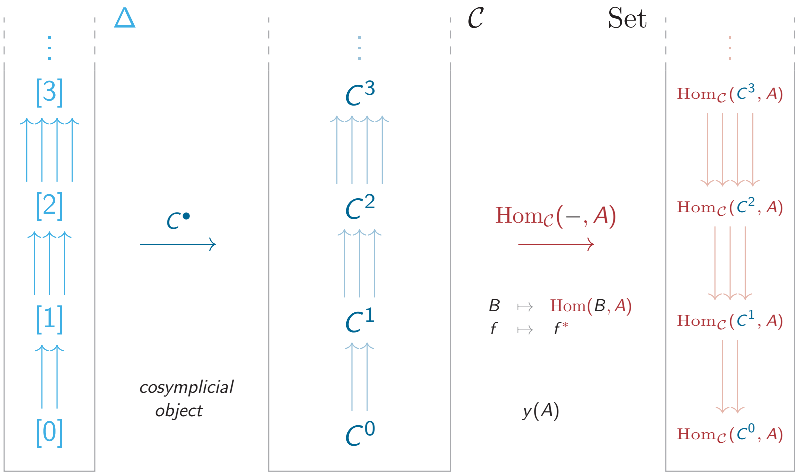
Definition

The nerve of A with respect to $C^\bullet$ is the simplicial set

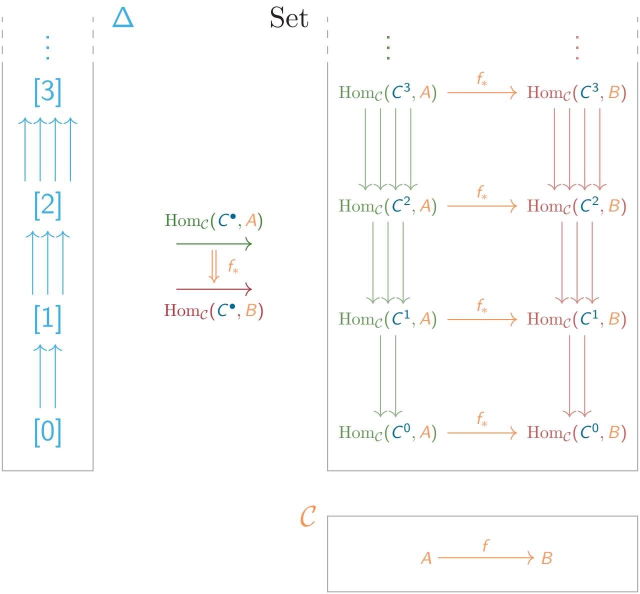
$$N_\bullet(A) = \mathrm{Hom}_{\mathcal{C}}(C^\bullet, A).$$

- The nerve defines a functor $N_\bullet : \mathcal{C} \to \mathrm{Set}_\Delta$.

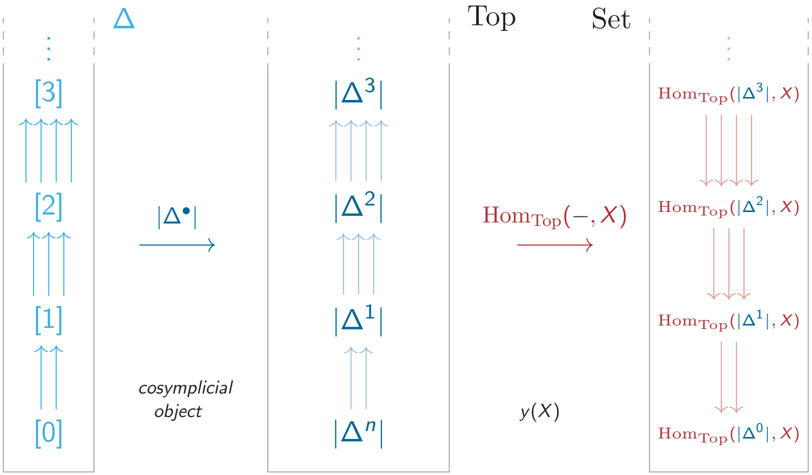
$$\begin{array}{ccc} A & \longrightarrow & N_\bullet(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ B & \longrightarrow & N_\bullet(B) \end{array}$$



$$N_\bullet(A) = \mathrm{Hom}_{\mathcal{C}}(C^\bullet, A)$$


Δ
Set
[3]
[2]
[1]
[0]
Hom_C(C^•, A)
f_*
Hom_C(C^•, B)
Hom_C(C^3, A)
Hom_C(C^3, B)
Hom_C(C^2, A)
Hom_C(C^2, B)
Hom_C(C^1, A)
Hom_C(C^1, B)
Hom_C(C^0, A)
Hom_C(C^0, B)
f_*
C
A
f
B

Example: singular simplicial complex



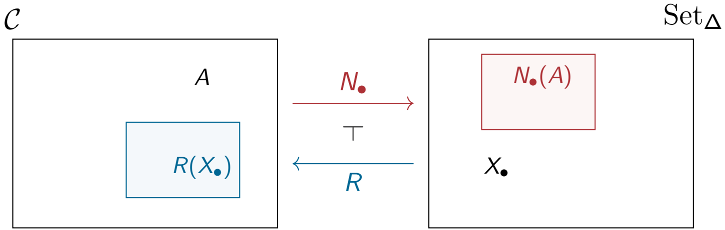
$$\mathrm{Sing}_\bullet(X) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^\bullet|, X)$$

Examples

category	cosimplicial object	nerve
Top	$\lvert\Delta^\bullet\rvert$	$\mathrm{Sing}_\bullet(X) = \mathrm{Hom}_{\mathrm{Top}}(\lvert\Delta^\bullet\rvert, X)$
Cat	inclusion	$N_\bullet(\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}([\bullet], \mathcal{C})$
$\mathrm{Ch}_{\geq 0}(\mathbb{Z})$	$N_*(\Delta^\bullet, \mathbb{Z})$	$K_\bullet(M_*) = \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(N_*(\Delta^\bullet, \mathbb{Z}), M_*)$
Δ	id_Δ	$[n] \mapsto \Delta^n$
Set	$[n] \mapsto \{*\}$	$X \mapsto \underline{X}$
Top-Cat	–	topological nerve
Kan-Cat	–	simplicial nerve
2-Cat	–	Duskin nerve

4. Realizations

Idea



$$\mathrm{Hom}_{\mathcal{C}}(R(X_\bullet), A) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(X_\bullet, N_\bullet(A))$$

Preliminary discussion

Recall that

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n_\bullet, X_\bullet) \underset{\textit{Yoneda}}{\cong} X_n$$

In particular,

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n_\bullet, N_\bullet(A)) \cong N_n(A) := \mathrm{Hom}_{\mathcal{C}}(C^n, A) .$$

So

$$\begin{aligned} r : y(\Delta) \subseteq \mathrm{Set}_\Delta &\longrightarrow \mathcal{C} \\ \Delta^n_\bullet &\longmapsto C^n \end{aligned}$$

satisfies

$$\mathrm{Hom}_{\mathcal{C}}(r(\Delta^n_\bullet), A) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n_\bullet, N_\bullet(A))$$

Facts from category theory

Let $\mathcal{D}$ be small and $\mathcal{C}$ cocomplete (i.e. small colimts exist)

1. Every $F \in \mathrm{Psh}(\mathcal{D})$ is a colimit of representables

$$F = \operatorname*{colim}_{y(D) \to F} y(D)$$

2. Every functor

$$r : y(\mathcal{D}) \to \mathcal{C}$$

has a colimit-preserving extension

$$\begin{aligned} R : \quad \mathrm{Psh}(\mathcal{D}) &\longrightarrow \mathcal{C} \\ \operatorname{colim} y(D) &\longmapsto \operatorname{colim} r\big(y(D)\big) \end{aligned}$$

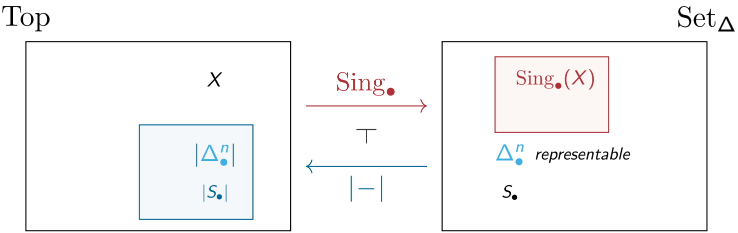
3. with right adjoint

$$\begin{aligned} N : \mathcal{C} &\to \mathrm{Psh}(\mathcal{D}) \\ A &\mapsto \mathrm{Hom}_{\mathcal{C}}(r(-), A) \end{aligned}$$

Application to Δ

1. Every $S_\bullet \in \mathrm{Set}_\Delta$ is a colimit of representables
$$F = \operatorname*{colim}_{\Delta^n_\bullet \to F} \Delta^n_\bullet$$
2. Every functor
$$r^\bullet : y(\Delta) \to \mathcal{C}$$
has a colimit-preserving extension
$$\begin{aligned} R : \quad \mathrm{Set}_\Delta &\longrightarrow \mathcal{C} \\ \operatorname{colim} \Delta^n_\bullet &\longmapsto \operatorname{colim} r(\Delta^n_\bullet) \end{aligned}$$
3. with right adjoint
$$\begin{aligned} N : \mathcal{C} &\to \mathrm{Set}_\Delta \\ A &\mapsto \mathrm{Hom}_{\mathcal{C}}(r^\bullet, A) \end{aligned}$$
"restriction of $\mathrm{Hom}_{\mathcal{C}}(-, A)$ along r"
4. Rearranging:
$$N_\bullet(A) = \mathrm{Hom}_{\mathcal{C}}(r^\bullet, A)$$

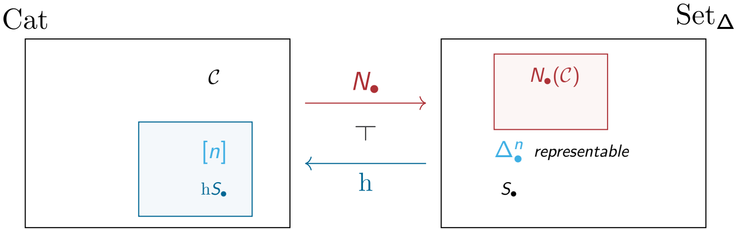
Example: geometric realization



$$\mathrm{Hom}_{\mathcal{C}}(|\Delta^n_\bullet|, X) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n_\bullet, \mathrm{Sing}_\bullet(X))$$

$$\mathrm{Hom}_{\mathcal{C}}(|S_\bullet|, X) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{Sing}_\bullet(X))$$

Example: homotopy category of a simplicial set



$$\mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n_\bullet, \mathrm{Sing}_\bullet(\mathcal{C}))$$

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}S_\bullet, \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{Sing}_\bullet(\mathcal{C}))$$

$$\mathrm{Ob}(\mathrm{h}S_\bullet) = S_0$$

$$\mathrm{Mor}(\mathrm{h}S_\bullet) = \langle S_1 \mid s_0 x = \mathrm{id}_x,\ d_1\sigma = d_0\sigma \circ d_2\sigma,\ x \in S_0, \sigma \in S_2 \rangle$$

$$s(e) = d_1 e,\ t(e) = d_0 e$$

5. Kan complexes

Horns

Fix $0 \leq i \leq n$.

Definition

The boundary $\partial\Delta^n \in \mathrm{Set}_\Delta$ of $\Delta^n_\bullet$ is given by

$$(\partial\Delta^n)_m = \left\{\alpha \in \Delta^n_m \ : \ \alpha \text{ is not surjective}\right\}$$

- $\partial\Delta \subseteq \Delta^n_\bullet$
- $\partial\Delta^n =$ "remove the interior of Δ^n "

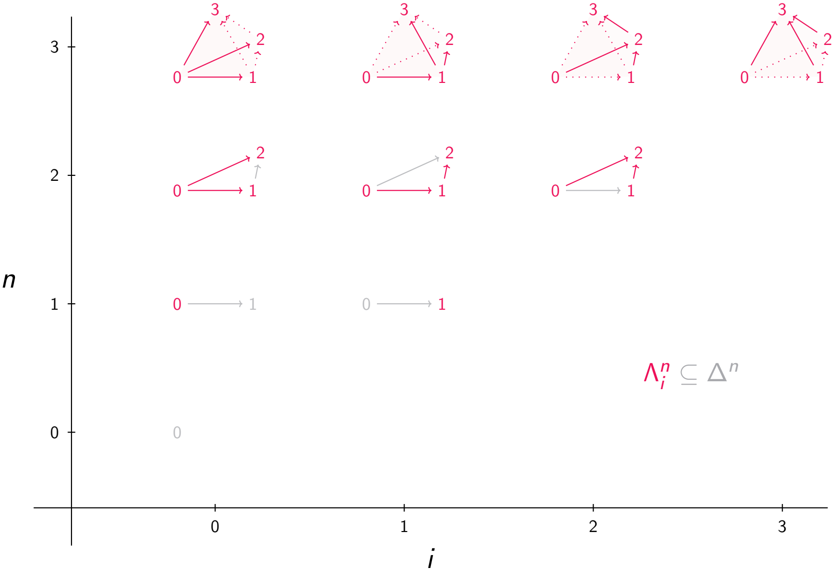
Definition

The ith horn $\Lambda^n_i \in \mathrm{Set}_\Delta$ of $\Delta^n_\bullet$ is given by

$$(\Lambda^n_i)_m = \left\{\alpha \in \Delta^n_m \ : \ [n] \not\subseteq \alpha([m]) \cup \{i\}\right\}$$

- $\Lambda^n_i \subseteq \partial\Delta^n$
- $\Lambda^n_i =$ "remove the ith face of $\partial\Delta^n$ "

Low-dimensional horns



Kan complexes

Definition

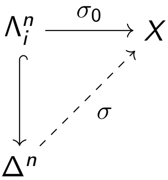
We say that $X_\bullet \in \mathrm{Set}_\Delta$ is a Kan complex if every simplicial map

$$\sigma_0 : \Lambda^n_i \to X_\bullet$$

extends to a simplicial map

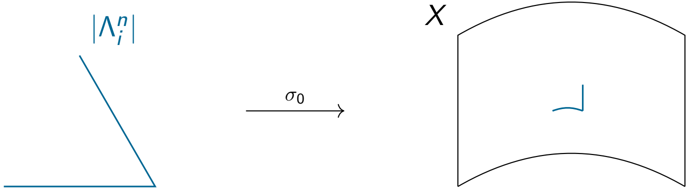
$$\sigma : \Delta^n \to X_\bullet$$

- "Every horn in $X_\bullet$ can be filled"
- i.e. $X_\bullet$ satisfies the horn-filling condition:

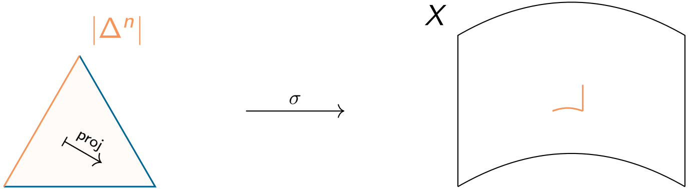


Example: $\mathrm{Sing}_\bullet(X)$

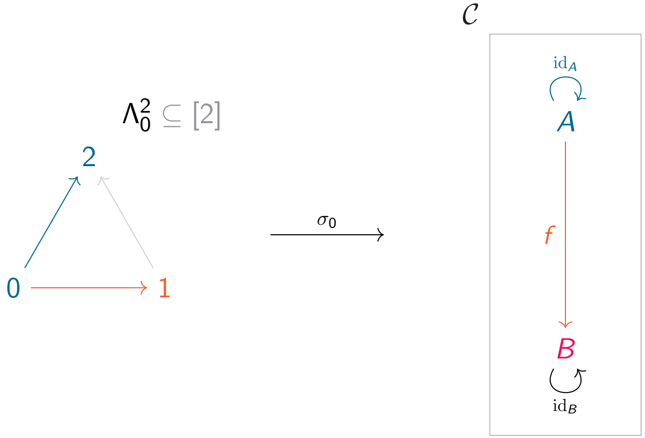
Every horn σ_0 *in* $\mathrm{Sing}_\bullet(X)$



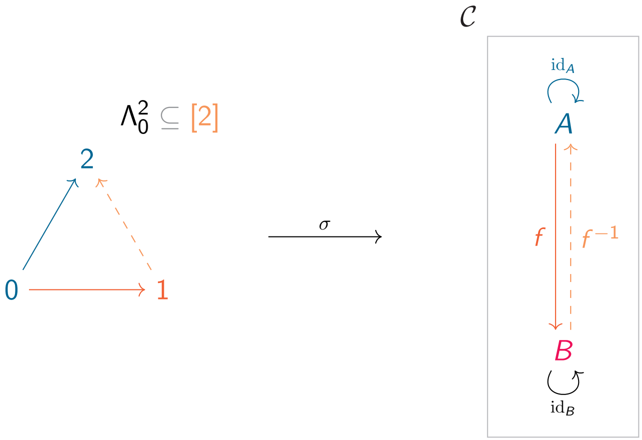
extends to a simplex σ *by projecting* $|\Delta^n|$ *to* $|\Lambda^n_i|$



Non-example: nerves of categories



Example: nerves of groupoids



Uniqueness of composition and inverses $\implies$ *uniqueness of extensions*