Research Statement

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I am interested in (multi)symplectic manifolds, Hamiltonian actions, and the geometry of moment maps. More recently, I have also become interested in higher quantization, (higher) stacks and gerbes, and higher geometry more generally. Broadly speaking, I am drawn towards mathematics that arises from physics.

My current research goals are to:

- investigate interactions between reduction and prequantization schemes in multisymplectic geometry,
- explore potential applications of this to quantum field theory and representation theory, and
- formulate and settle an extension of the $[Q, R] = 0$ "quantization commutes with reduction" theorem to the multisymplectic setting.

I have also been interested in interactions between information-theoretic measures of entanglement and geometric quantization, and in infinitesimal variations of generalized complex reduced spaces.

In previous work, I have:

- developed a theory of polysymplectic geometry for the study of moduli spaces of flat connections [12, 14],
- computed the symplectic volume of special classes of these spaces [12],
- introduced an approach to polysymplectic quantization and proved that the Guillemin–Sternberg $[Q, R] = 0$ theorem does not extend to the polysymplectic setting [14],
- formulated and proved a multisymplectic reduction theorem, as well as an associated multisymplectic Duistermaat–Heckman theorem [15], and
- in collaboration with Antonio Miti and Leonid Ryvkin, proposed a reduction scheme for the L_∞-algebra of observables associated to a premultisymplectic manifold, which also yields a new symplectic reduction scheme for the Poisson algebra of smooth functions on a symplectic manifold [16].

In the remainder of this document, I will first describe my active research interests and then survey my previous work.

1 *Current Interests:* higher quantization and reduction

By *higher quantization* I have in mind the higher gerbe quantization of multisymplectic manifolds, following the framework of Fiorenza–Rogers–Schreiber [29] as well as the earlier quantization scheme of Rogers [57]. These procedures are naturally approached as analogues of symplectic geometric quantization in the general n-plectic and 2-plectic settings, respectively.

See Section 2 for relevant background on n-plectic geometry.

1.1 Background on symplectic quantization

In the setting of symplectic manifolds, *quantization* refers to an assignment to each symplectic manifold (M, ω) with certain additional data, of a Hilbert space $\mathcal{H}$. When (M, ω) is equipped with a compatible G-action, the space $\mathcal{H}$ is a linear representation of G.

There are at least two motivating perspectives on symplectic quantization:

1. *As a model for physical systems.* The Hilbert space of quantum states $\mathcal{H}$ represents a physical quantum system and the symplectic manifold (M, ω) encodes the associated classical space[9]. A distinguished Hamiltonian function $h : M \to \mathbb{R}$ induces a quantum evolution of $\mathcal{H}$ in line with the Schrödinger equation, and ω yields a measure $\|\sigma\|^2 \frac{1}{n!}\omega^n \in \Omega^{\mathrm{top}}(M)$ according to which a measurement of the quantum state $\sigma \in \mathcal{H}$ outputs a classical state $p \in M$.

   Additional data is needed to determine the space $\mathcal{H}$, the classical space (M, ω) is insufficient. A *polarization*, i.e. an local identification of M as a generalized momentum phase space, must be introduced. This is not only – and certainly not the most popular – approach to modeling physical quantization.

2. *As an approach to representation theory.* According to this view, a symplectic manifold (M, ω) is heuristically the classical counterpart to a Hilbert space of associated quantum states $\mathcal{H}$, and a Hamiltonian G-action on (M, ω) realizes $\mathcal{H}$ as a unitary G-representation. This approach is developed in the *orbit method* [37], which investigates certain irreducible representations $\mathcal{H}$ arising as the quantization of the natural action of G on the canonically-symplectic coadjoint orbits $(\mathcal{O}, \omega_{\mathrm{can.}})$ of $\mathfrak{g}^*$.

Each perspective is interesting and invites further investigation in the multisymplectic setting.

1.2 Stacks and gerbes

Informally, a *stack* is a higher fiber bundle and a *gerbe with band G* is a higher G-principal bundle. There are multiple distinct constructions that realize this idea. We can identify three broad approaches:

1. *Sheaf-like constructions* that associate categories to the open neighborhoods of a smooth manifold M in a locally-sensitive manner [19, 45], analogous to the way in which sheaves associate sets (i.e. 0-categories) to the open neighborhoods of M. This approach is explicitly categorical, and invokes the theory of 2-categories. This is conceptually similar to constructions in terms of fibrations over the category of smooth manifolds [10].

2. *Combinatorial constructions*, which define a gerbe on M in terms of local "lower" data taking the form of admissible local principal bundles and suitably compatible gluing maps [20, 32].

3. *Bundle-like constructions* that associate pointwise data to M, specifically the fibers of a surjective submersion $X \to M$ with a bundle defined on the total space of the fiber product $X^{[2]} \to M$, together with additional data and conditions that are themselves pointwise in terms of M [46, 47].

The the term "higher" may be taken to reference higher category theory: Where a 0-stack (i.e. a bundle) is characterized in terms of 1-categorical limits of admissible local data on M, a 1-stack satisfies a condition of 2-categorical limits.

The transition from bundles to stacks forms the first step in a hierarchy of constructions given in the language of n-categories, as exposited, for example, by Nikolaus–Schreiber–Stevenson [50, 51]. This, in turn, is grounded in the work of Lurie [41, 40] on developing the *quasicategory* approach to higher category theory due to Joyal [35], which is itself a development of the *restricted Kan complexes* of Boardman–Vogt [18].

1.3 Research goals

Regarding higher quantization, my guiding objectives are to:

- investigate interactions between reduction and prequantization schemes in multisymplectic geometry,
- explore potential applications of this to quantum field theory and representation theory, and
- formulate and settle an extension of the $[Q, R] = 0$ "quantization commutes with reduction" theorem to the multisymplectic setting.

My intent is to follow the approach of Fiorenza–Rogers–Schreiber [28] and Rogers [57]. The relevant constructions are:

Definition ([28])**.** The cochain complex of sheaves

$$C^\infty(-;U(1)) \xrightarrow{\mathrm{dlog}} \Omega^1(-) \xrightarrow{\mathrm{d}} \Omega^2(-) \xrightarrow{\mathrm{d}} \cdots \Omega^n(-) \to \Omega^{n+1}(-) \to \cdots,$$

with $C^\infty(-;U(1))$ in degree 0, will be called the *Deligne complex* and will be denoted by the symbol $\underline{U}(1)_{\mathrm{Del}}$.

Definition ([28])**.** The *n-stack of principal $U(1)$-n-bundles (or $(n-1)$-bundle gerbes) with connection* $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is the n-stack presented via the Dold–Kan construction to the presheaf $\underline{U}(1)^{\leq n}_{\mathrm{Del}}[n]$ regarded as a presheaf of chain complexes concentrated in nonnegative degree.

Definition ([28])**.** Let (M, ω) be a pre-n-plectic manifold. A *prequantization* of (M, ω) is a lift

$$\begin{tikzcd} & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[d, "F"] \\ M \arrow[ur, dashed, "\nabla"] \arrow[r, "\omega"'] & \Omega^{n+1}(-)_{\mathrm{cl}} \end{tikzcd}$$

The key observation here is that this prequantization construction is locally defined in terms of geometric data for which my collaborators and I have recently introduced a very general reduction scheme [16] (see Subsection 2.3).

In the presence of a subset $N \subseteq M$ and compatible action $\mathfrak{g} \curvearrowright M$, the definition of a *reduced prequantization* suggests itself as:

Definition. The *reduced Deligne complex* is

$$C^\infty(-;U(1))_N \xrightarrow{\mathrm{d}} \Omega^1(-)_N \xrightarrow{\mathrm{d}} \Omega^2(-)_N \xrightarrow{\mathrm{d}} \cdots \Omega^n(-)_N \to \Omega^{n+1}(-)_N \to \cdots$$

and the *reduced n-stack of principal $U(1)$-n-bundles with connection* $\mathbf{B}^n U(1)_{\mathrm{conn},N}$ is the n-stack associated to $\underline{U}(1)^{\leq n}_{\mathrm{Del}}[n]_N$ by the Dold–Kan correspondence.

Definition. We say that $\nabla : M \to \mathbf{B}^n U(1)$ is a *reducible prequantization* if the associated Čech–Deligne cocycle consists of reducible forms.

Definition. The *reduced prequantization* $\nabla_N : M_N \to \mathbf{B}^n U(1)_N$ is the image of ∇ in $\mathbf{B}^n U(1)_N$.

Here (M_N, ω_N) denotes the multisymplectic reduction of $\mathfrak{g} \curvearrowright (M, \omega)$ along N, as defined in my collaboration [16], modeled on the procedure proposed in my previous work [15] (see Subsection 2.2).

Intriguingly, Fiorenza–Rogers–Schreiber also establish that the L_∞-algebra of observables $\mathrm{Ham}_\infty(M,\omega)$ is suitably isomorphic to space of infinitesimal quantormorphisms of any quantization of the multisymplectic manifold (M,ω) [28]. In light of this, it would be interesting to clarify the relation between our recent L_∞-reduction procedure and the quantum reduction procedure proposed above.

2 Multisymplectic geometry

The main construction is the following.

Definition. An *n-plectic manifold* is a smooth manifold M equipped with a closed nondegenerate form $\omega \in \Omega^{n+1}(M)$.

For example, a 1-plectic manifold is precisely a symplectic manifold.

Multisymplectic geometry arises naturally in physics. The space of classical fields in a given field theory, realized as the sections of a particular configuration bundle $E \to \Sigma^n$, yields an associated *multimomentum bundle* $\Lambda^n_1 E \to E$ of 1-horizontal n-forms on the total space E. The total space $\Lambda^n_1 E$ possesses a canonical multisymplectic structure ω generalizing the canonical symplectic structure on T^*Q of symplectic geometry. Accordingly, we obtain a correspondence:

$$\text{infinite-dimensional spaces of fields} \underset{\text{de-transgression}}{\overset{\text{transgression}}{\rightleftarrows}} \text{finite-dimensional multimomentum bundles.}$$

A “translation dictionary” is given by:

physics	**mathematics**
classical fields	multisymplectic manifolds
quantum-field symmetries	L_∞-algebras of observables
quantum fields	higher quantizations

In addition to potential applications to classical field theory and field quantization, multisymplectic geometry is interesting in its own right. In comparison with symplectic geometry, the general multisymplectic stetting is extremely flexible and identifying underlying structure and patterns is more difficult.

2.1 Background on symplectic reduction

Informally, a *reduction* scheme is a systematic procedure that takes a geometric structure X equipped with symmetries, and returns a *reduced* structure $X_{\text{red.}}$ which is in some sense “smaller” than X. The prototypical example of reduction is the Marsden–Weinstein–Meyer symplectic reduction theorem, as follows.

Theorem (Marsden–Weinstein ’74, Meyer ’73)**.** *Let $G \curvearrowright (M,\omega)$ be a Hamiltonian action and let $\mu : M \to \mathfrak{g}^*$ be a moment map. If $\mu^{-1}(\lambda) \subseteq M$ is smooth and $G_\lambda \curvearrowright \mu^{-1}(\lambda)$ is free and proper, then there is a unique symplectic form $\omega_\lambda \in \Omega^2(M_\lambda)$ such that $\pi^*\omega_\lambda = i^*\omega$.*

Devising a reduction procedure requires a good understanding of both the nature of symmetries and of how these symmetries can be expressed in terms of the underlying geometric structure. In the symplectic setting, this is realized in terms of a *moment map* $\mu : M \to \mathfrak{g}^*$, which expresses the infinitesimal symmetries $G \curvearrowright (M,\omega)$ in terms of the Poisson algebra of classical observables $C^\infty(M,\omega)$.

2.2 Multisymplectic reduction

The question of multisymplectic reduction has been open and of interest for at least several decades.

> *Reduction theory is by no means completed. . . . Only a few instances and examples of multisymplectic reduction are really well understood. . . so one can expect to see more activity in this area as well.*

— J. Marsden and A. Weinstein, 2001, *Comments on the history, theory, and applications of symplectic reduction*

> *One of the most interesting problems in multisymplectic geometry is how to extend the well-known Marsden–Weinstein reduction scheme for symplectic manifolds to the case of multisymplectic structures.*

— M. de León, 2018, Review of "Remarks on multisymplectic reduction" by Echeverría-Enríquez, Muñoz-Lecanda and Román-Roy

I propose such a reduction scheme in [15]. I define a *moment map* for an n-plectic action $G \curvearrowright (M, \omega)$ to be a particular form $\mu \in \Omega^{n-1}(M, \mathfrak{g}^*)$, a *reduction parameter* to be any closed form $\phi \in \Omega^{n-1}(M, \mathfrak{g}^*)$, the *$\phi$-level set* to be the equalizer

$$\mu^{-1}(\phi) = \{p \in M \mid \mu_p = \phi_p\}$$

and the *reduced space* to be the quotient $M_\phi = \mu^{-1}(\phi)/G_\phi$ by the ϕ-isotropy subgroup $G_\phi \subseteq G$. The main result is:

Theorem 1 ([15]). *If $\mu^{-1}(\phi) \subseteq M$ is smooth and $G_\phi \curvearrowright \mu^{-1}(\phi)$ is free and proper, then there is a unique closed $(k+1)$-form ω_ϕ on M_ϕ such that $\pi^*\omega_\phi = i^*\omega$.*

In addition to this, under restrictive conditions I prove a formula on the variation of the cohomology class of the reduced form ω_ϕ, modeled on the classical symplectic formula of Duistermaat–Heckman [26].

2.3 L_∞-reduction

> *Śniatycki and Weinstein have defined an algebraic reduction in the context of group actions and moment maps which is guaranteed to produce a reduced Poisson algebra but not necessarily a reduced space of states.*

— J. Stasheff, 1997, *Homological reduction of constrained Poisson algebras*

Background

Given a symplectic Hamiltonian system (M, ω, G, μ) and a dual Lie algebra element $\lambda \in \mathfrak{g}^*$, the classic Marsden–Weinstein symplectic reduction theorem provides a reduced symplectic manifold $(M_\lambda, \omega_\lambda)$ when $\mu^{-1}(\lambda) \subseteq M$ is smooth and when $G_\lambda \curvearrowright \mu^{-1}(\lambda)$ is free and proper. This yields a reduction map

$$r_\lambda : C^\infty(M)^G \to C^\infty(M_\lambda)$$

from the Poisson subalgebra of G-invariant functions – i.e. classical observables commuting with the components of the moment map $\mu : M \to \mathfrak{g}^*$ – on (M, ω) to the Poisson algebra of smooth functions on the reduced space $(M_\lambda, \omega_\lambda)$. In this case, we could say that $C^\infty(M_\lambda)$ is the *reduction* of the Poisson algebra of classical observables on (M, ω).

It is an interesting fact that, even when the conditions on (M,ω,G,μ) and $\lambda \in \mathfrak{g}^*$ are not met and the reduced space $(M_\lambda,\omega_\lambda)$ does not exists, it is sometimes still possible to define a *reduced Poisson algebra of observables* $C^\infty(M,\omega)_\lambda$ that generalizes the construction of $C^\infty(M_\lambda,\omega_\lambda)$. Various definitions for this reduced space have been propose, including Dirac [25], Śniatycki–Weinstein [61], Arms–Gotay–Jennings [4], and Arms–Cushman–Gotay [3] reduction.

In contrast with the symplectic setting, an n-plectic manifold (M,ω) does not generally possess an associated Poisson algebra of observables. Rather, it carries an *L_∞-algebra* of observables $\mathrm{Ham}_\infty(M,\omega) \cong \Omega^{\leq n-1}(M)$. An L_∞-algebra is the higher instantiation of a Lie algebra. It may be realized as a graded algebra $\oplus_{i\in\mathbb{N}} L_i$ equipped with a family of alternating k-ary brackets $\{l_k\}_{k\in\mathbb{N}}$ satisfying a compatibility condition that generalizes the Jacobi property of a Lie bracket $[\,,]$.

As with the Poisson algebra $C^\infty(M,\omega)$ in the symplectic case, the L_∞-algebra of observables $\mathrm{Ham}_\infty(M,\omega)$ is equivalent to the space of infinitesimal automorphisms of any prequantization of (M,ω). Thus, as discussed in Subsection 1.3, a reduction scheme for $\mathrm{Ham}_\infty(M,\omega)$ is a natural precursor to a full reduction scheme for multisymplectic (pre)quantizations.

From Poisson to L_∞-reduction

As a symplectic manifold yields an associated Poisson algebra of classical observables $C^\infty(M,\omega)$, so a multisymplectic manifold carries an associated L_∞-algebra of observables $\mathrm{Ham}_\infty(M,\omega)$ [58].

In [16] my collaborators and I propose a general reduction procedure for the L_∞-algebra of observables $\mathrm{Ham}_\infty(M,\omega)$ associated to a multisymplectic manifold (M,ω), in such a way that reflects the reduction of the Poisson algebra of observables induced by the Marsden–Weinstein symplectic reduction.

Our construction is defined to be the quotient of the L_∞-subalgebra $\mathrm{Ham}_\infty(M,\omega)_{[N]}$ of *reducible observables* by the L_∞-ideal $I_{\mathrm{Ham}_\infty}(N)$ of observables that vanish along N.

Definition 3.21 ([16])**.** The *reduction* of $\mathrm{Ham}_\infty(M,\omega)$ with respect to $\mathfrak{g} \curvearrowright (N \subseteq M)$ is the L_∞-algebra

$$\mathrm{Ham}_\infty(M,\omega)_N = \frac{\mathrm{Ham}_\infty(M,\omega)_{[N]}}{I_{\mathrm{Ham}_\infty}(N)} \ .$$

There are two salient features:

1. There are no smooth or topological condition imposed on the subset $N \subseteq M$. For example, we may even take $N = \mathbb{Q}$ and $M = \mathbb{R}$.
2. The reduced L_∞-algebra of observables $\mathrm{Ham}_\infty(M,\omega)_N$ is not guaranteed to be of the form $L_\infty(M_N,\omega_N)$ for any premultisymplectic manifold (M_N,ω_N).

The first main result of our paper is that the quotient $\mathrm{Ham}_\infty(M,\omega)_N$ does indeed inherit an L_∞-algebra structure from $\mathrm{Ham}_\infty(M,\omega)$. The second main result is that this algebraic reduction naturally embeds in the L_∞-algebra of observables $\mathrm{Ham}_\infty(M_N,\omega_N)$ associated to the geometric reduced space (M_N,ω_N), whenever the smooth premultisymplectic manifold (M_N,ω_N) exists.

Theorem 3.38. *The geometric reduction map*

$$\begin{aligned} r_N : \mathrm{Ham}_\infty(M,\omega)_{[N]} &\to \mathrm{Ham}_\infty(M_N,\omega_N) \\ (v,\alpha) &\mapsto (v_N,\alpha_N) \\ \alpha &\mapsto \alpha_N \end{aligned}$$

is a strict L_∞-morphism with kernel $I_{\mathrm{Ham}_\infty}(N)$. In particular, there is a natural inclusion of

L_∞*-algebras*

$$\mathrm{Ham}_\infty(M,\omega)_N = \frac{\mathrm{Ham}_\infty(M,\omega)_{[N]}}{I_{\mathrm{Ham}_\infty}(N)} \xhookrightarrow{\bar{r}_N} \mathrm{Ham}_\infty(M_N,\omega_N).$$

A further interesting feature regarding our reduction scheme is that, when (M,ω) is symplectic, the L_∞-reduced space $C^\infty(M,\omega)_N$ is a Poisson algebra. Thus, we introduce a new symplectic reduction scheme for the Poisson algebra $C^\infty(M,\omega)$.

3 Polysymplectic geometry

While attempting to understand the moduli space of flat connections over a general manifold, I arrived at a theory of *vector-valued symplectic geometry* [13]. This independently-developed theory turns out to be equivalent in the finite dimensional setting to an earlier *polysymplectic formalism* of Günther [31], the *k-symplectic formalism* of Awane [6], the *generalized symplectic geometry* of Norris [52], and the theory of *p-almost cotangent structures* of de León, Méndez, and Salgado [24]. Most of these formalisms were designed to provide a new geometric framework for physical field theories.

Polysymplectic geometry can also be viewed as a unifying framework for Hamiltonian and admissible bi-Hamiltonian systems $(M,\{\,,\}_0,\{\,,\}_1)$. A bi-Hamiltonian system is a distinguished family of Poisson structures $s\{\,,\}_0+t\{\,,\}_1$, $s,t\in\mathbb{R}$, with applications to the study of integrable systems [27].

A similarity may further be drawn between polysymplectic manifolds and bi-Hermitian manifolds (M,g,I_+,I_-) by considering the 2-forms $\omega_\pm = g(\,\cdot\,,I_\pm\cdot\,)$. These form a special case of generalized complex manifolds[30], which were introduced by Hitchin [33] and arise in the study of mirror symmetry [11], and interact with twistor theory [38, 55, 59], which was originally developed by Penrose as an approach to quantum gravity [53].

3.1 V-symplectic manifolds

Fix a manifold M and vector space V.

Definition ([13])**.** A *V-symplectic structure* on M is a closed V-valued 2-form $\omega\in\Omega^2(M,V)$ which is nondegenerate in the sense that $\iota_X\omega=0$ only if $X=0$.

Examples include:

1. The semisimple Lie group G with $\mathfrak{g}$-symplectic structure $-\mathrm{d}\theta$, and $\theta\in\Omega^1(G,\mathfrak{g})$ is the Maurer-Cartan form on G. The model space is $\mathfrak{g}$, with linear $\mathfrak{g}$-symplectic structure given by the Lie bracket.

2. The phase space $\mathrm{Hom}(TQ,V)$ with V-symplectic structure $-\mathrm{d}\theta$, and θ is given by $\theta_\phi(X)=\phi(\pi_*X)$. The model space is $U\oplus\mathrm{Hom}(U,V)$, with linear V-symplectic structure $\omega(u+\phi,u'+\phi')=\phi'(u)-\phi(u')$, where U is the model space of Q.

It is interesting to observe that certain longstanding open problems in symplectic geometry are quickly settled in the polysymplectic case, for example, the following.

Arnold Conjecture (See [44], Chapter 11)**.** *A symplectomorphism that is generated by a time-dependent Hamiltonian vector field should have at least as many fixed points as a Morse function on the manifold must have critical points.*

Theorem 3.21 ([13])**.** *The Arnold conjecture fails in the V-symplectic setting.*

A counterexample is provided by left multiplication on the $\mathfrak{g}$-symplectic manifold $(G,-\mathrm{d}\theta)$. In my thesis [12], I obtained the following reduction theorem.

V-Symplectic Reduction Theorem. *Let (M, ω, G, μ) be a V-Hamiltonian systems and fix $\alpha \in \mathrm{Hom}(\mathfrak{g}, V)$. If the stabilizer subgroup G_α of α under the coadjoint action is connected, and if $M_\alpha = \mu^{-1}(\alpha)/G_\alpha$ is smooth, then there is a unique V-valued 2-form $\omega_\alpha \in \Omega^2(M_\alpha, V)$ such that*

$$\pi^*\omega_\alpha = i^*\omega,$$

where $i : \mu^{-1}(\alpha) \hookrightarrow M$ is the inclusion and $\pi : \mu^{-1}(\alpha) \to M_\alpha$ is the projection. The form ω_α is closed and is nondegenerate at πx if and only if $\underline{\mathfrak{g}_\alpha}_x = \underline{\mathfrak{g}}_x^{\omega\omega} \cap \underline{\mathfrak{g}}_x$.

Unlike the symplectic case, the reduced V-valued 2-form may be degenerate. It should be noted that this result appeared earlier in [42].

3.2 Applications to gauge theory

The aim of [13] is to exhibit the moduli space of flat connections over a manifold of arbitrary dimension as the polysymplectic reduction of the space of all connections. This extends an observation of Atiyah and Bott [5] in the case that M is a closed orientable surface.

Theorem 4.12 ([13])**.** *Let M be a compact manifold of dimension at least 3, G a compact matrix Lie group, P a G-principal bundle on M with connected gauge group $\mathcal{G}$, $\mathcal{A}$ the space of connections on P, and $k > \frac{1}{2}\dim M + 1$ a fixed integer. Denote the the $W^{k,2}$ Sobolev completion of $\mathcal{A}$ by $\mathcal{A}_k$, and likewise for $\mathcal{G}$, $\mathscr{g}$, and Ω^*, and write $\tilde{\Omega}^2(M)$ and $\tilde{B}^2(M)$ for the spaces of C^1 forms and coboundaries on M, respectively. Let $F : \mathcal{A}_k \to \Omega^2_{k-1}(M, \mathrm{ad}P)$ be the curvature. The function*

$$\mu : \mathcal{A}_k \to \mathrm{Hom}\big(\mathscr{g}_{k+1}, \tilde{\Omega}^2(M)/\tilde{B}^2(M)\big),$$

given by

$$\mu(A)(f) = \langle F_A \wedge f\rangle_{\tilde{\Omega}^2/\tilde{B}^2}, \qquad f \in \Omega^0_{k+1}(M, \mathrm{ad}P) \cong \mathscr{g}_{k+1},$$

is a moment map for the action of $\mathcal{G}_{k+1}$ on $\mathcal{A}_k$ with respect to the polysymplectic structure $\omega \in \Omega^2\big(\mathcal{A}, \tilde{\Omega}^2(M)/\tilde{B}^2(M)\big)$, defined by

$$\omega(\alpha, \beta) = \langle \alpha \wedge \beta\rangle_{\tilde{\Omega}^2/\tilde{B}^2},$$

for $\alpha, \beta \in \Omega^1_k(M, \mathrm{ad}P) \cong T_A\mathcal{A}_k$. The reduced space at 0 coincides with the moduli space of flat connections $\mathcal{M}_k = F^{-1}(0)/\mathcal{G}_{k+1}$ on P. On the smooth points of $\mathcal{M}_k$, the reduced 2-form ω_0 takes values in the second cohomology $H^2(M)$.

A similar characterization holds for the space of generalized Ricci flat connections on a holomorphic vector bundle E over a complex manifold M.

Definition 4.21 ([13])**.** We call the connection $A \in \mathcal{A}(E)$ *Ricci flat* if $\mathrm{tr}\, F_A = 0$.

Corollary 4.22 ([13])**.** *Let M be a compact complex manifold and let E be a holomorphic vector bundle over M with $c_1(E) = 0$. The moduli space of Ricci flat connections is the polysymplectic reduction of the space of connections $\mathcal{A}_k(E)$.*

3.3 V-symplectic quantization

By lifting the Hamiltonian dynamics of a V-symplectic manifold (M, ω) to the space of sections of a Hermitian vector bundle $E \to M$, in [14] I arrive at a natural definition of V-symplectic prequantization. This approach, which identifies quantization as an extension of symmetries, reflects the early work of Souriau [62] in the symplectic setting.

A comparison with the symplectic case yields,

symplectic	V-symplectic
prequantum line bundle (L, ∇)	prequantum vector bundle (E, ∇, A)
scalar multiplication m	effective unitary representation $A : V \to \operatorname{End} E$
Planck's constant $\hbar > 0$	weights of A
prequantum operator $Q_f = \nabla_{X_f} + \mathrm{i}\hbar m_f$	$Q_f = \nabla_{X_f} + A_f$
curvature condition $F^\nabla = -\mathrm{i}\hbar\, \omega$	$F^\nabla = -A_\omega$

The fundamental construction is as follows.

Definition 4.1 ([14])**.** Let (M, ω) be a V-symplectic manifold, $\mathcal{O} \subseteq C^\infty_H(M, V)$ an algebra of classical observables, and η an invariant measure on M. A *prequantization* of $(M, \omega, \mathcal{O}, \eta)$ consists of a Hermitian vector bundle $E \to M$ and a faithful first-order Lie algebra representation

$$Q : \mathcal{O} \to \operatorname{End} \Gamma(M, E),$$

which preserves the inner product on the subspace of smooth L^2 sections of $\Gamma(M, E)$ with respect to η, and which extends the Hamiltonian vector fields on M in the sense that

$$Q_f(s\psi) = (X_f s)\psi + sQ_f\psi,$$

for all $f \in \mathcal{O}$, $s \in C^\infty(M)$, and $\psi \in \Gamma(E)$. By *first-order*, we mean that if $\mathrm{d}f$ vanishes at $x \in M$ then $(Q_f\psi)_x = 0$ for all $\psi \in \mathcal{H}$.

Under the assumption that M is *transitive*, namely that the algebra of classical observables generates the tangent bundle, every prequantization is realized by a *prequantum vector bundle*, that is, a faithful Hermitian V-module bundle $E \to M$ with compatible unitary connection ∇ satisfying $F^\nabla = -\omega$. Subject to this constraint, the following classification theorem is obtained.

Definition 4.11 ([14])**.** A full lattice $I \subseteq V$ is called a *prequantum lattice* for (M, ω) if $[\omega]_{H^2(M,V)}$ lies in the image of $H^2(M, I) \hookrightarrow H^2(M, V)$, that is, if the pairing $\langle \omega, \cdot \rangle : H_2(M, \mathbb{Z}) \to V$ takes values in I.

Theorem 4.13 ([14])**.** *If (M, ω) is transitive and connected, then there is a natural correspondence between equivalence classes of minimal rank prequantizations of (M, ω) and bases of prequantum lattices $I \subseteq V$ for (M, ω).*

The main result of [14] is that the Guillemin–Sternberg $[Q, R] = 0$ theorem does not extend to the polysymplectic setting.

Theorem 5.16 ([14])**.** *Let (E, ∇, A) be a positive definite prequantum vector bundle on (M, ω, J, G, μ) and suppose that (M_0, ω_0) is nonempty and V-symplectic, and inherits a complex structure J_0 and prequantum vector bundle E_0. It is not generally the case that $\mathcal{H}_J(M)_G \cong \mathcal{H}_{J_0}(M_0)$.*

In addition to this, I define a notion of V-symplectic spin$^\mathrm{c}$ quantization, which could in turn be interesting to compare with [36].

4 The volume of the moduli space of flat connections

The primary aim of [12] is to compute the symplectic volume of the moduli space of flat connections over manifolds of arbitrary dimension. While the moduli space $\mathcal{M}_G(M)$ is smoothly equivalent to the character variety $\operatorname{Hom}(\pi_1(M), G)/G$, the latter does not generally exhibit a natural symplectic structure, and thus the volume cannot be obtained by means of this equivalence. When it exists, the volume of a symplectic manifold (M^{2n}, ω) is conventionally defined to be

$$\operatorname{vol} M = \int_M \frac{\omega^n}{n!}.$$

Up to rescaling, ω^n represents the unique measure on M which is preserved by all symplectic transformations. It may be physically motivated in light of its relation to certain canonical measures arising in statistical mechanics. The factor $1/n!$ may be motivated by the observation that $\omega^n/n!$ is identified with the standard volume form on $\mathbb{R}^{2n}$ by any symplectic coordinate chart.

4.1 Symplectic volume and the space of quantum states

Suppose (L, ∇) is a positive prequantum line bundle over the Kähler manifold (M^{2n}, ω). By this we will mean that L is a positive Hermitian line bundle on M with connection ∇ and curvature $2\pi\mathrm{i}\,\omega \in \Omega^2(M, \mathbb{C}) \cong \Omega^2(M, \operatorname{End} L)$. For each $k > 0$, the kth tensor power $(L^k, k\nabla)$ is a prequantum line bundle for the rescaled symplectic manifold $(M, k\omega)$. The curvature condition ensures that L^k is holomorphic and the Riemann-Roch formula provides that

$$\sum_{i=0}^{2n} (-1)^i \dim H^i(M, L^k) = \int_M \operatorname{ch} L^k \wedge \operatorname{Td} M.$$

On the left-hand side, the Kodaira vanishing theorem yields $H^i(M, L^k) = 0$ for $i > 0$ and $k \gg 0$. On the right-hand side, we have $\operatorname{ch} L^k = (k\omega)^n/n! + O(k^{n-1})$ and $\operatorname{Td} M = 1 + \Omega^{\geq 2}(M)$. Thus, in the large k limit,

$$\dim H^0(M, L^k) = k^n \operatorname{vol} M + O(k^{n-1}).$$

In sum, the symplectic volume describes the growth rate of the quantum state space $H^0(M, L^k)$ in the semiclassical limit $\hbar = \frac{2\pi}{k} \to 0$.

4.2 The computation of the volume

The moduli space of flat connections over a closed surface Σ is naturally a symplectic manifold. It arises, for example, as the space of dynamical solutions in classical Chern-Simons theory. In this situation, Σ represents a spacelike slice of a 3-dimensional spacetime. By the preceding discussion, the volume of the moduli space describes the growth rate of the quantum state space as the energy of the system tends to infinity.

Verlinde [65] obtained the volume using techniques from conformal field theory. Witten [66] later provided a direct combinatorial proof. Liu [39] employed a heat kernel argument in the context of a compact oriented surface with prescribed holonomy around punctured disks. Ho and Jeffrey [34] addressed the nonorientable case.

In my PhD thesis [12] I compute the volume of the moduli space $\mathcal{M}_G(M)$ over a manifold M of arbitrary dimension, subject to conditions on M and G. First, for a general manifold M and an abelian structure group G.

Theorem 10.1 ([12])**.** *Let M be a symplectic (resp. Riemannian) manifold and let T be a compact abelian Lie group equipped with an invariant metric. Then*

$$\operatorname{vol} \mathcal{M}_T(M) = \operatorname{vol}(T)^{b_1(M)} \operatorname{vol} H^1(M, \mathbb{Z}) \left|\operatorname{Hom}(H_1(M, \mathbb{Z})_{Tor}, T)\right|,$$

where $\operatorname{vol} H^1(M, \mathbb{Z})$ denotes the covolume of the lattice $H^1(M, \mathbb{Z}) \subseteq H^1(M, \mathbb{R})$ with respect to the symplectic (resp. Riemannian) structure on M.

Second, for a manifold M with free abelian fundamental group, and a general structure group G. This setting is very closely related to spaces of commuting elements in G, a topic of substantial recent interest [56, 1, 22, 49, 54, 7, 23, 63, 2, 8].

Theorem 10.2 ([12])**.** *Let M be a symplectic (resp. Riemannian) manifold with free abelian fundamental group $\pi_1(M)$, G a compact connected semisimple Lie group of dimension k and rank ℓ, $\langle\,,\rangle$ an* Ad*-invariant metric on the Lie algebra $\mathfrak{g}$, W the Weyl group of G with respect to some maximal* *torus,* $\{\alpha\} \subseteq H^*$ *the root system, and* $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ *the half sum of a subsystem of positive roots. Then*

$$\operatorname{vol}\mathcal{M}_G(M) = \Big(\frac{\operatorname{vol}G}{\sqrt{2\pi}^{\,k-\ell}}\prod_{\alpha>0}\alpha\rho\Big)^{b_1(M)}\frac{1}{|W|}\operatorname{vol}H^1(M,\mathbb{Z}),$$

where $\operatorname{vol}H^1(M,\mathbb{Z})$ *denotes the covolume of the lattice* $H^1(M,\mathbb{Z})$ *in* $H^1(M,\mathbb{R})$*.*

The key observation in each case is that $\operatorname{Hom}(\pi_1(M),G)$ is smoothly equivalent to $\operatorname{Hom}(H^1(M),G)$ under the given restrictions. This yields a tractable model for the tangent fibers of $\mathcal{M}_G(M)$ in terms of $H^1(M)$ and the adjoint representation of G.

Setting $G = U(1)$ yields the following.

Corollary 10.2 ([12])**.** *The moduli space of complex line bundles over a manifold* M *with flat connection has volume*

$$\operatorname{vol}\mathcal{M}_{U(1)}(M) = (2\pi)^{b_1(M)}\operatorname{vol}H^1(M,\mathbb{Z})\,\big|\mathrm{Ch}(H_1(M,\mathbb{Z})_{Tor})\big|,$$

where $\mathrm{Ch}(H_1(M,\mathbb{Z})_{Tor})$ *is the set of characters of* $H_1(M,\mathbb{Z})_{Tor}$*.*

4.3 Immersions of the moduli space

The moduli space over a surface $\mathcal{M}_G(\Sigma)$ has been extensively studied. In [12], I show that under suitable conditions the restriction of connections over M to a surface $\Sigma \subseteq M$ forms a symplectic immersion of $\mathcal{M}_G(M)$ in $\mathcal{M}_G(\Sigma)$.

Theorem 11.2 ([12])**.** *If* $n \geq 2$*, then there is a compact, connected embedded surface* $\Sigma \subseteq M$ *such that* $[\Sigma] \in H^2(M)$ *is the Poincaré dual of* η*. The inclusion* $i : \Sigma \hookrightarrow M$ *yields a symplectic immersion* $i^* : \mathcal{M}_G(M) \to \mathcal{M}_G(\Sigma)$*. At a connection* A *on* M*, the codimension of the image is equal to*

$$\dim\ker\big(\,H^2_A(M,\Sigma;\operatorname{ad}\mathfrak{g}) \to H^2_A(M,\operatorname{ad}\mathfrak{g})\,\big).$$

5 Eigenvalues of the p-Laplacian

In [17], S. Seto and I obtained a lower bound for the first eigenvalue of the p-Laplacian on a Kähler manifold. The p-Laplacian is a nonlinear differential operator given by

$$\Delta_p(f) = \operatorname{div}\big(|\nabla f|^{p-2}\nabla f\big),$$

and describes a diffusion process $\partial_t f = \Delta_p f$ with diffusivity equal to a power of the speed $|\nabla f|$. When $p = 2$, it is the ordinary Laplacian. The eigenvalue condition is given by

$$\Delta_p(f) = -\mu|f|^{p-2}f.$$

When M is closed, the first eigenvalue satisfies the following variational characterization,

$$\mu_{1,p} = \inf\left\{\frac{\int_M|\nabla f|^p}{\int_M|f|^p} \mid f \in W^{1,p}(M)\backslash\{0\}, \int_M |f|^{p-2}f = 0\right\}.$$

When $\partial M \neq \emptyset$ and we impose Dirichlet boundary conditions,

$$\lambda_{1,p} = \inf\left\{\frac{\int_M|\nabla f|^p}{\int_M|f|^p} \mid f \in W^{1,p}_c(M)\backslash\{0\}\right\}.$$

Matei [43], Valtorta [64], and Naber and Valtorta [48] obtained lower estimates on the first eigenvalue under different assumptions on the Ricci curvature. Seto and Wei [60] computed various lower bounds in terms of the integral Ricci curvature. Chen and Wei [21] obtained upper estimates on submanifolds of space forms.

In [17], we specialize to Kähler manifolds.

Theorem 1.1 ([17])**.** *Let (M, J, g) be an $n = 2m$ (real) dimensional Kähler manifold, possibly with boundary. Assume that the underlying (real) Ricci curvature satisfies* $\mathrm{Ric} \geq Kg$ *for some constant* $K > 0$*. If $\partial M = \emptyset$, then for $p \geq 2$,*

$$\mu_{1,p}^{\frac{2}{p}} \geq \frac{p+2}{p(p-1)}K = \left(1 + \frac{2}{p}\right)\frac{K}{p-1}.$$

If $\partial M \neq \emptyset$, we assume the convexity condition that $\frac{p}{2}H + \mathrm{II}(J\mathbf{n}, J\mathbf{n}) \geq 0$ and the Dirichlet boundary condition, where $\mathbf{n}$ is the unit outward normal vector field on ∂M, H is the mean curvature, and II *is the second fundamental form. Then for $p \geq 2$,*

$$\lambda_{1,p}^{\frac{2}{p}} \geq \frac{p+2}{p(p-1)}K.$$

In the course of the proof we also establish the following p-Reilly formula.

Lemma 2.2 ([17])**.** *For $f \in C^2(M)$ and $p \geq 2$,*

$$\begin{aligned}
&\int_{\partial M} |\nabla f|^{p-2}\big\{ -(\Delta_{\partial M} f + H\nabla_n f)\nabla_n f - \mathrm{II}(\nabla_{\partial M} f, \nabla_{\partial M} f) + \langle \nabla(\nabla_n f), \nabla f\rangle_{\partial M}\big\} \\
&\qquad = (p-2)\int_M |\nabla f|^{p-2}\big|\nabla|\nabla f|\big|^2 - \int_M (\Delta f)(\Delta_p f) \\
&\qquad\quad + \int_M |\nabla f|^{p-2}\big(2|H_2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + \langle \mathrm{Hess} f, J^*\mathrm{Hess} f\rangle\big).
\end{aligned}$$

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