Reduction of multisymplectic manifolds

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Outline

1. Symplectic reduction
2. Multisymplectic reduction
3. The Duistermaat–Heckman theorem

Based on:

B., Reduction of multisymplectic manifolds, *Lett. Math. Phys.*, 2021

The problem of multisymplectic reduction

> *Reduction theory is by no means completed. . . . Only a few instances and examples of multisymplectic reduction are really well understood. . . so one can expect to see more activity in this area as well.*

— J. Marsden and A. Weinstein, 2001, *Comments on the history, theory, and applications of symplectic reduction*

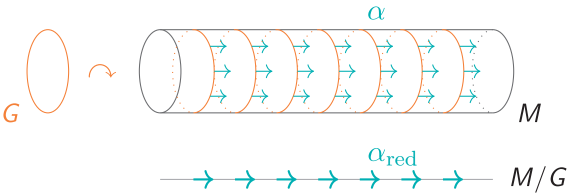
> *One of the most interesting problems in multisymplectic geometry is how to extend the well-known Marsden–Weinstein reduction scheme for symplectic manifolds to the case of multisymplectic structures.*

— M. de León, 2018, *Review of "Remarks on multisymplectic reduction" by Echeverr´ia-Enr´iquez, Muñoz-Lecanda, and Román-Roy*

The action descent lemma

If

- $G \curvearrowright M$ free and proper,
- $\alpha \in \Omega^*(M)$ *invariant* and *horizontal* ($\iota_{\underline{\mathfrak{g}}}\alpha = 0$),



then

- $\exists!\, \alpha_{\mathrm{red}} \in \Omega^*(M/G)$ such that $\alpha = \pi^*\alpha_{\mathrm{red}}$,
- $\mathrm{d}\alpha = 0 \implies \mathrm{d}\alpha_{\mathrm{red}} = 0$.

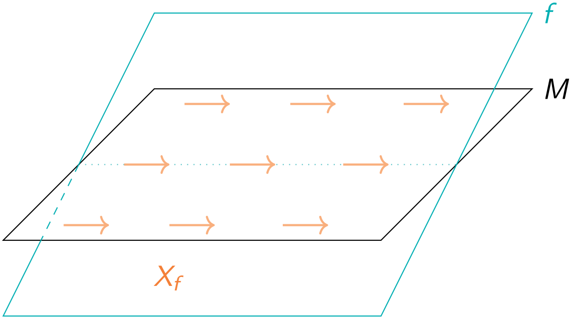
1. Symplectic reduction

Symplectic Hamiltonian dynamics

observables $\longrightarrow$ symmetries

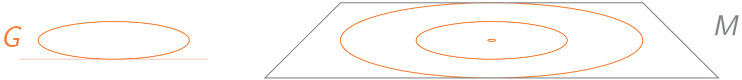
$C^\infty(M) \ni f$ $\qquad$ $X \in \mathfrak{X}(M)\,, \quad \mathcal{L}_X\omega = 0$

$$\mathrm{d}f = \iota_{X_f}\omega$$

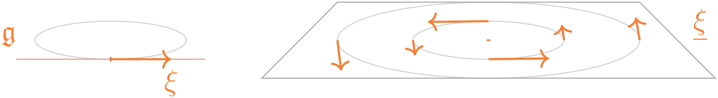


Symplectic Hamiltonian actions

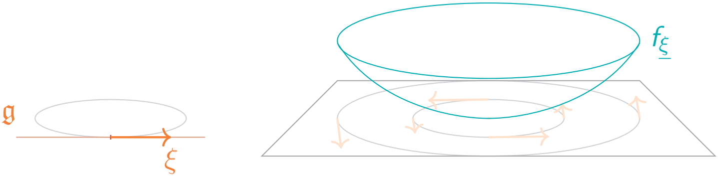
To specify a symplectic action $G \curvearrowright M$ . . .



we could describe the induced map $\xi \mapsto \underline{\xi}$ . . .



or the assignment of Hamiltonian functions $\xi \mapsto f_{\underline{\xi}}$.



When this is possible,* the action is called **Hamiltonian**.

*and $\xi \mapsto f_{\underline{\xi}}$ is a homomorphism of Lie algebras

Symplectic Hamiltonian G-spaces

moment map: Describe $G \curvearrowright M$ *in terms of* $C^\infty(M) \to \mathfrak{X}(M)$.

$$
\begin{array}{ccccc}
 & & C^\infty(M) & & f \\
 & \overset{\langle \mu, \cdot \rangle}{\nearrow} & \downarrow & & \downarrow \\
\mathfrak{g} & \longrightarrow & \mathfrak{X}(M) & & X_f \\
\xi & \longmapsto & \underline{\xi} & &
\end{array}
$$

as Lie algebras

i.e., μ_ξ generates $\underline{\xi}$

moment map: $\mu : M \to \mathfrak{g}^*$

Hamiltonian G-space: (M, ω, G, μ)

Symplectic reduction

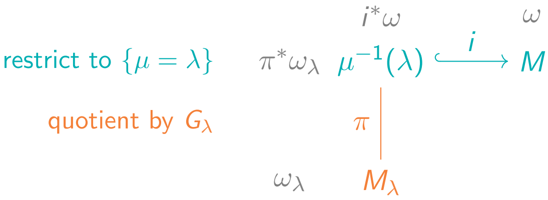
Two ingredients:

1. Hamiltonian G-space (M, ω, G, μ)
2. parameter $\lambda \in \mathfrak{g}^*$

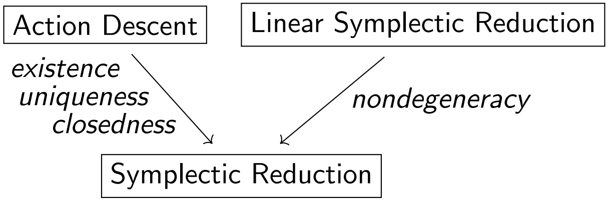
The *reduced space* is $M_\lambda := \mu^{-1}(\lambda)/G_\lambda$.

Theorem (Marsden–Weinstein '74, Meyer '73)

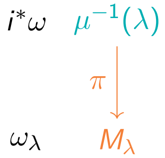
If $\mu^{-1}(\lambda) \subseteq M$ is smooth and $G_\lambda \curvearrowright M$ is free and proper, then there is a unique symplectic form $\omega_\lambda \in \Omega^2(M_\lambda)$ *such that $\pi^*\omega_\lambda = i^*\omega$.*



Symplectic reduction — proof



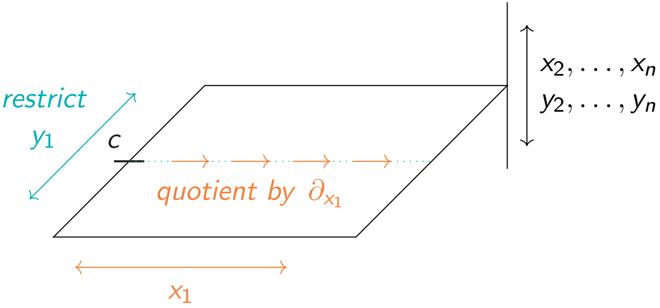
1. Apply the *Action Descent Lemma* to $G_\lambda \curvearrowright \mu^{-1}(\lambda)$ and $i^*\omega$.



2. Use *Linear Symplectic Reduction* to conclude that ω_λ is nondegenerate.

Symplectic reduction — idea

$$\omega = \underbrace{\mathrm{d}x_1 \wedge \mathrm{d}y_1}_{\text{to be removed}} + \mathrm{d}x_2 \wedge \mathrm{d}y_2 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$$



Restrict and quotient conjugate degrees of freedom.

Heuristic approach to reduction

1. describe $G \curvearrowright M$ in terms of ω

   moment map μ

2. identify a distinguished reduced space

   reduction at $\mu = 0$

3. use the ambiguity in 1. to obtain a family of reduced spaces

   reduction at $\mu - \lambda = 0$,
   i.e. reduction at $\mu = \lambda$

Note: If $G_\lambda \neq G$, then $\mu - \lambda : M \to \mathfrak{g}^*$ is not a moment map for either $G \curvearrowright M$ or $G_\lambda \curvearrowright M$.

2. Multisymplectic reduction

Multisymplectic Hamiltonian dynamics

observables	$\longrightarrow$	symmetries
~~$C^\infty(M) \ni f$~~		$X \in \mathfrak{X}(M)\ , \quad \mathcal{L}_X\omega = 0$
$\Omega^{k-1}_H(M) \ni \alpha$		

$$\mathrm{d}\alpha = \iota_{X_\alpha}\omega$$

Hamiltonian vector fields are indeed multisymplectic symmetries:

$$\begin{aligned} \mathcal{L}_{X_\alpha}\omega &= \mathrm{d}\iota_{X_\alpha}\omega + \iota_{X_\alpha}\mathrm{d}\omega \\ &= \mathrm{d}\mathrm{d}\alpha \\ &= 0 \end{aligned}$$

Two brackets on $\Omega^{k-1}_H(M)$

1. $\{\alpha, \beta\} = \mathcal{L}_{X_\alpha}\beta$ ← ***Leibniz bracket*** (Jacobi, but not antisymmetric)
2. $\{\alpha, \beta\}' = \iota_{X_\alpha}\mathrm{d}\beta = \iota_{X_\alpha}\iota_{X_\beta}\omega$

Equal up to a coboundary:

$$\mathcal{L}_{X_\alpha}\beta = \iota_{X_\alpha}\mathrm{d}\beta + \mathrm{d}\iota_{X_\alpha}\beta$$

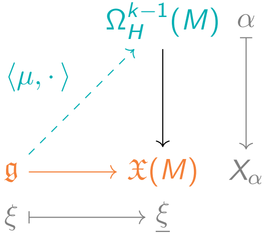
$\alpha \mapsto X_\alpha$ is a homomorphism of Leibniz algebras:

$$\mathrm{d}\{\alpha, \beta\} = \mathrm{d}\mathcal{L}_{X_\alpha}\beta = \mathcal{L}_{X_\alpha}\iota_{X_\beta}\omega = \iota_{[X_\alpha, X_\beta]}\omega$$

$$\implies \quad X_{\{\alpha,\beta\}} = [X_\alpha, X_\beta]$$

Multisymplectic Hamiltonian G-spaces

moment map: Describe $G \curvearrowright M$ *in terms of* ~~$C^\infty(M)$~~ $\Omega_H^{k-1}(M) \to \mathfrak{X}(M)$.



as Leibniz algebras

i.e., μ_ξ generates $\underline{\xi}$

moment map: $\mu \in \Omega^{k-1}(M, \mathfrak{g}^*)$

Hamiltonian G-space: (M, ω, G, μ)

The moment map conditions

- $G \curvearrowright M$
- $\mu \in \Omega^{k-1}(M, \mathfrak{g}^*)$

1. Hamiltonian condition:

$$\mathrm{d}\mu_\xi = \iota_\xi \omega$$

2. Leibniz condition:

$$\mu_{[\xi,\zeta]} = \{\mu_\xi, \mu_\zeta\}$$

equivalently,

$$\mathcal{L}_\xi \mu_\zeta = \mu_{[\xi,\zeta]}$$

The space of moment maps

- (M, ω, G, μ) Hamiltonian G-space
- $\phi \in \Omega^{k-1}(M, \mathfrak{g}^*)$

Question: When is $\mu + \phi$ a moment map?

- $\mathrm{d}\phi = 0$, since

$$\mathrm{d}(\mu + \phi)_\xi = \iota_\xi \omega \iff \mathrm{d}\phi_\xi = 0.$$

- $\mathcal{L}_\xi \phi_\zeta = \phi_{[\xi,\zeta]}$, as

$$\mathcal{L}_\xi(\mu + \phi) = (\mu + \phi)_{[\xi,\zeta]} \iff \mathcal{L}_\xi \phi = \phi_{[\xi,\zeta]}.$$

i.e. ϕ is a moment map for the trivial action $G \curvearrowright M$.

The space of moment maps is an affine space modeled on $\{\phi \in \Omega^{k-1}(M, \mathfrak{g}^*) \mid \mathrm{d}\phi = 0,\ G_\phi = G\}$.

Level sets of the moment map

Rather than:

- family of moment maps $\{\mu - \phi \,|\, \mathrm{d}\phi = 0,\ G_\phi = G\}$
- reduction at $\mu - \phi = 0$

We instead consider:

- fixed moment map μ
- family of levels $\{\phi \,|\, \mathrm{d}\phi = 0,\ \cancel{G_\phi = G}\}$
- reduction at $\mu = \phi$

ϕ-level set:

$$\mu^{-1}(\phi) := \{\mu = \phi\}$$

Multisymplectic reduction

Two ingredients:

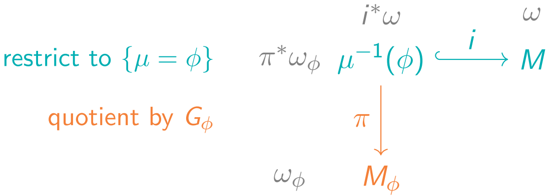
1. Hamiltonian G-space (M, ω, G, μ)
2. parameter $\phi \in \Omega^{k-1}(M, \mathfrak{g}^*)$ with $\mathrm{d}\phi = 0$

Define the *reduced space*,

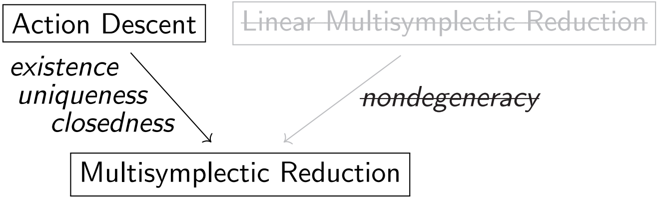
- $M_\phi = \mu^{-1}(\phi)/G_\phi$

Theorem (B. '20)

If $\mu^{-1}(\phi) \subseteq M$ *is smooth and* $G_\phi \curvearrowright \mu^{-1}(\phi)$ *is free and proper, then there is a* unique closed $(k+1)$-form ω_ϕ *on* M_ϕ *such that* $\pi^*\omega_\phi = i^*\omega$.



Multisymplectic reduction — proof idea



1. Apply the *Action Descent Lemma* to $G_\phi \curvearrowright \mu^{-1}(\phi)$ and $i^*\omega$.

$$\begin{array}{cc} i^*\omega & \mu^{-1}(\phi) \\ & \downarrow \pi \\ \omega_\phi & M_\phi \end{array}$$

2. ~~Use *Linear Multisymplectic Reduction* to conclude that ω_ϕ is nondegenerate.~~

Extension: reduction of closed forms

1. The proof makes no use of the nondegeneracy or homogeneity of $\omega \in \Omega^{k+1}(M)$.
2. Extends naturally to a reduction scheme for closed forms.

$$\omega \in \Omega^*(M) \text{ closed}$$
$$\phi \in \Omega^*(M, \mathfrak{g}^*) \text{ closed}$$

$$\frac{\mu \in \Omega^*(M, \mathfrak{g}^*)}{\begin{aligned} \mathrm{d}\mu_\xi &= \iota_\xi \omega \\ \mathcal{L}_\xi \mu_\zeta &= \mu_{[\xi,\zeta]} \end{aligned}}$$

3. The Duistermaat–Heckman theorem

Associated results

Duistermaat–Heckman theorem

- locally polynomial dependence of $\mathrm{vol}\, M_\lambda$ on $\lambda \in \mathfrak{t}^*$
- exact stationary phase approximation
- inverse Fourier transform of the pushforward of the Liouville measure by μ

(Infinitessimal) Duistermaat–Heckman theorem

- (M, ω, T, μ) Hamiltonian T-space
- $C \subseteq \mathfrak{t}^*$ open, with $T \curvearrowright \mu^{-1}(C)$ free

Question: How does $\omega_\tau \in \Omega^2(M_\tau)$ depend on $\tau \in C$?

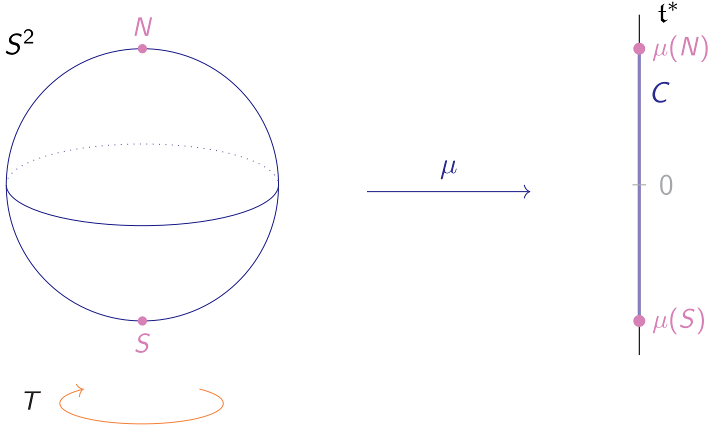
Theorem (Duistermaat–Heckman '82)

For each $\tau \in C$,

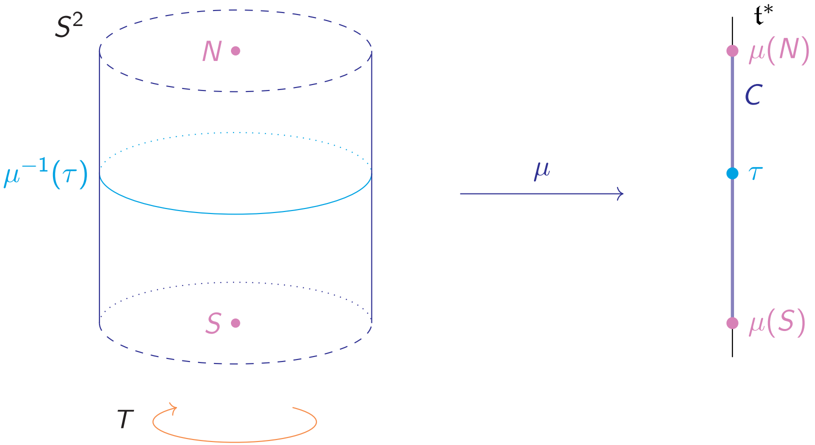
$$\partial_\lambda [\omega_\tau] = \langle c, \lambda \rangle, \qquad \lambda \in \mathfrak{t}^*,$$

where $c \in H^2(M_\tau, \mathfrak{t})$ is the Chern class of $T \curvearrowright \mu^{-1}(\tau)$.

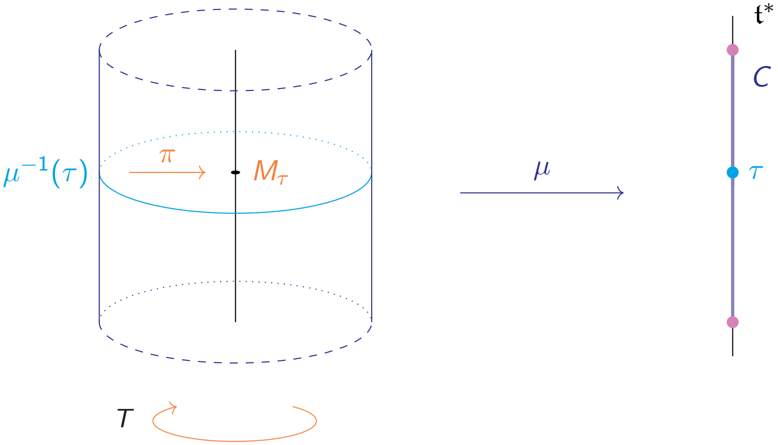
$$\begin{array}{ccc} T \curvearrowright \mu^{-1}(\tau) & \overset{i}{\hookrightarrow} & M \\ \downarrow \pi & & \\ M_\tau & & \end{array}$$



T acts freely on $\mu^{-1}(C)$.

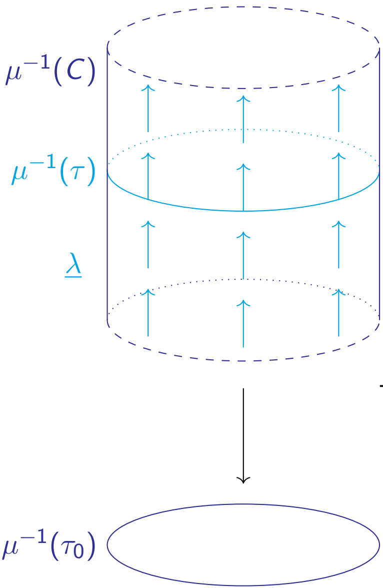


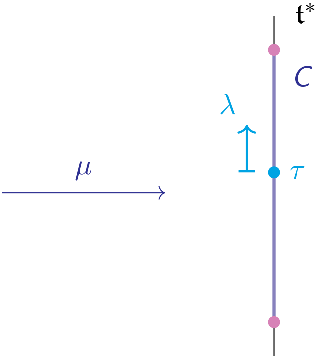
T preserves $\mu^{-1}(\tau)$.



$\mu^{-1}(\tau) \to M_\tau$ is a T-principal bundle.

$\mu^{-1}(C) \to C$ is a *family* of T-principal bundles.

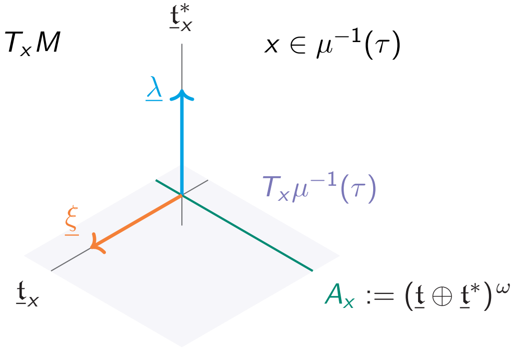




Trivialization $\mu^{-1}(C) \cong C \times \mu^{-1}(\tau_0)$

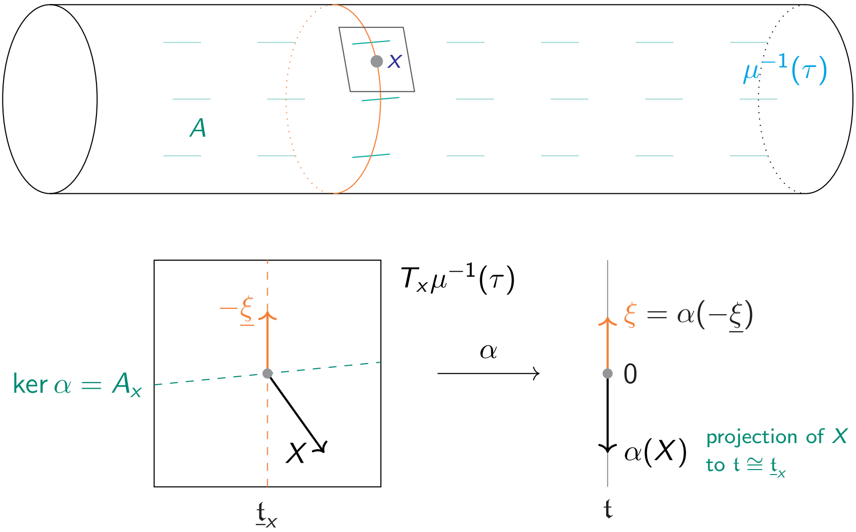
$\implies$ lift $\lambda \mapsto \underline{\lambda} \in \mathfrak{X}(\mu^{-1}(C))$

$\implies$ fundamental distribution $\underline{\mathfrak{t}}^*$



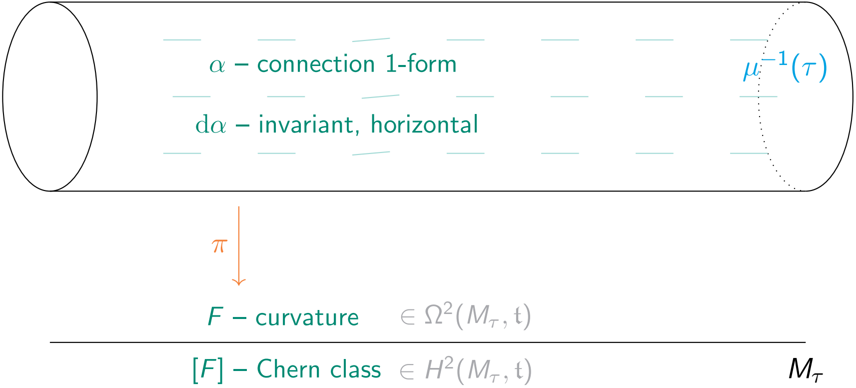
$$\begin{cases} T\mu^{-1}(\tau) = \underline{\mathfrak{t}} \oplus A \\ A \text{ is } T\text{-invariant} \end{cases} \implies \quad A \text{ is a connection on } \mu^{-1}(\tau) \to M_\tau$$

$A = 0$ in the previous illustrations



Connection 1-form: T-invariant $\alpha \in \Omega^1(\mu^{-1}(\tau), \mathfrak{t})$ with $\alpha(-\underline{\xi}) = \xi$.

Here we are using $\xi_x := \frac{\partial}{\partial t}\, e^{-t\xi} \cdot x \,\big|_{t=0}$.



$c = [F]$ does not depend on α:

- $\alpha - \alpha'$ is invariant and horizontal
- $F - F' = \mathrm{d}(\alpha - \alpha')_{\mathrm{red}}$

The key identity

$$\begin{aligned} \pi^*\partial_\lambda\omega_\tau &= \partial_\lambda\pi^*\omega_\tau, && \text{since } \underline{\lambda} \text{ is } T\text{-basic,} \\ &= \partial_\lambda i^*\omega, && \text{by the defining condition } \pi^*\omega_\tau = i^*\omega, \\ &= i^*\mathcal{L}_\lambda\omega, && \text{since } \underline{\lambda} \in \mathfrak{X}\big(\mu^{-1}(C)\big) \text{ lifts } \lambda \in \mathfrak{X}(C), \\ &= i^*\mathrm{d}\iota_\lambda\omega, && \text{as } \mathcal{L}_\lambda = \mathrm{d}\iota_\lambda + \iota_\lambda\mathrm{d}, \\ &= \mathrm{d}\, i^*\iota_\lambda\omega \end{aligned}$$

"co-connection 1-form":

$$\begin{aligned} \tilde{\alpha} : \mathfrak{t}^* &\xrightarrow{\langle\alpha,\cdot\rangle} \Omega^1\big(\mu^{-1}(\tau)\big), \\ \lambda &\longmapsto i^*\iota_\lambda\omega \end{aligned}$$

connection 1-form: $\alpha \in \Omega^1\big(\mu^{-1}(\tau), \mathfrak{t}\big)$.

$$\langle\mathcal{L}_\xi\alpha, \lambda\rangle = \mathcal{L}_\xi i^*\iota_\lambda\omega = i^*\iota_{[\lambda,\xi]}\omega = 0$$

$$\langle\alpha(\underline{\xi}), \lambda\rangle = \iota_\xi i^*\iota_\lambda\omega = -i^*\iota_\lambda\mathrm{d}\mu_\xi = \langle\xi, \lambda\rangle$$

$$\begin{aligned} \pi^*\partial_\lambda \omega_\tau &= \mathrm{d}\, i^* \iota_\lambda \omega, && \text{from the previous slide,} \\ &= \mathrm{d}\langle \alpha, \lambda \rangle, && \text{by the definition of } \alpha, \\ &= \pi^* \langle F, \lambda \rangle, && \text{since } \pi^* F = \mathrm{d}\alpha. \end{aligned}$$

$$\implies \quad \partial_\lambda \omega_\tau = \langle F, \lambda \rangle, \qquad \text{by the action descent lemma } (\exists !),$$

$$\implies \quad \partial_\lambda [\omega_\tau] = \langle c, \lambda \rangle$$

□

A multisymplectic extension

Goal: Extend the D–H theorem to the multisymplectic setting.

We need a distribution on M that plays the role of $\underline{\mathfrak{t}}^*$ in the symplectic setting.

Conjugate subspaces

- (E, ω) multisymplectic vector space

Definition

Subspaces $U, V \subseteq E$ are *conjugate* if

$$\begin{aligned} U \times V &\to \Lambda^{k-1} E^* \\ (X, Y) &\mapsto \iota_Y \iota_X \omega \end{aligned}$$

is nondegenerate and of rank 1.

In this case, we say that any nonzero element $\eta \in \Lambda^{k-1} E^*$ in the image of this map *conjugates* U and V.

Conjugate distributions

- (M, ω) multisymplectic manifold

Definition

Distributions $U, V \subseteq TM$ are

- *conjugate* if there is a closed form $\eta \in \Omega^{k-1}(M)$ which conjugates the fibers of U and V at every point of M.
- *strongly conjugate* if there is a 2-form $\sigma \in \Omega^2(M)$ such that $\iota_Y \iota_X \omega = \sigma(X, Y)\,\eta$ for $X \in U_x$ and $Y \in V_x$ at every $x \in M$.

σ defines bundle isomorphisms

$$U \xrightarrow{\sim} V^*$$

and

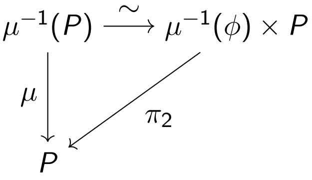
$$V \xrightarrow{\sim} U^*$$

Multisymplectic D–H theorem — hypotheses

- $\phi \in \Omega^{k-1}(M, \mathfrak{t}^*)$ closed and T-invariant
- $\mu^{-1}(\phi) \subseteq M$ embedded submanifold with free action of T
- $\eta \in \Omega^{k-1}(M)$ closed and T-invariant
- $C \subseteq \mathfrak{t}^*$ open

Assume that:

1. for $P = C \wedge \eta + \phi$, the diagram



   trivializes $\mu : \mu^{-1}(P) \to P$ as a family of T-principal bundles modeled on $\mu^{-1}(\phi)$,

2. $\exists\, \underline{\mathfrak{t}}^* \subseteq TM$ strongly conjugate to $\underline{\mathfrak{t}}$ by η.

Multisymplectic D–H theorem

Theorem (B. '20)

Fix $\psi \in P$. Under the hypotheses of the previous slide, we have

$$\partial_\lambda [\omega_\psi] = \langle c, \lambda \rangle \wedge [\eta_\psi], \qquad \lambda \in \mathfrak{t}^*,$$

where

- $c \in \Omega^2(M_\phi, \mathfrak{t})$ *is the Chern class of the model space* $\mu^{-1}(\phi) \to M_\phi$,
- η_ψ *is the reduction of* η *to* M_ψ.

Key identity:

$$\pi^* \partial_\lambda \omega_\psi = \mathrm{d}\, \underbrace{i^* \iota_\lambda \sigma}_{\langle \alpha, \lambda \rangle} \wedge \eta$$

Thank you!