Quantization of polysymplectic manifolds

Casey Blacker

Euler International Mathematical Institute, *and*
Saint Petersburg State University

Young Researchers' Virtual Multisymplectic Geometry Conference
13 July 2021

Outline

1. Geometric prequantization
2. Polysymplectic manifolds
3. Polysymplectic prequantization

Based on:

B., Quantization of polysymplectic manifolds, *J. Geom. Phys.*, 2019

Aim

In this brief presentation, my aim is to motivate the following construction.

Definition (prequantum vector bundle)

A *prequantum vector bundle* (E, ∇, A) on a V-symplectic manifold (M, ω) consists of a

1. Hermitian vector bundle $E \to M$,
2. fiberwise V-module structure $A : V \to \mathrm{End}\, E$,
3. unitary connection ∇ on E with $\nabla A = 0$,

such that the curvature $F^{\nabla} \in \Omega^2(M, \mathrm{End}\, E)$ satisfies

$$F^{\nabla} = -A_{\omega}.$$

1. Geometric prequantization

Hamiltonian symmetries

$$\begin{array}{ccc} \text{functions} & \longrightarrow & \text{symmetries} \\ C^\infty(M) \ni f & & X \in \mathfrak{X}(M)\,, \quad \mathcal{L}_X\omega = 0 \end{array}$$

Definition

The *Hamiltonian vector field* $X \in \mathfrak{X}(M)$ associated to a function $f \in C^\infty(M)$ is defined by

$$\mathrm{d}f = -\iota_X\omega.$$

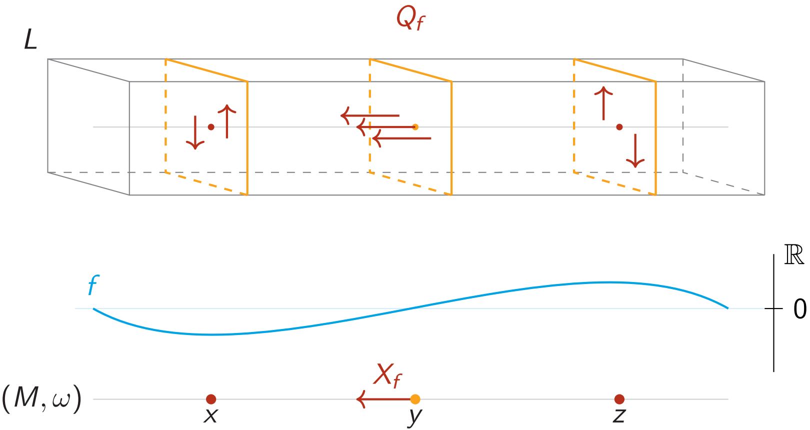
Conversely, f is called a *Hamiltonian function* of X.

The *Poisson bracket* on $C^\infty(M)$ is given by

$$\{f, h\} = X_f h$$

Idea

Extend the symmetries of (M,ω) to the space of sections of a Hermitian line bundle $L \to M$.



Definition

A prequantization of a symplectic manifold (M, ω) consists of

i. a Hermitian line bundle $L \to M$,
ii. a unitary connection ∇ on L with curvature $F_\nabla = \mathrm{i} c\, \omega$, for some nonzero constant $c \in \mathbb{R}$.
iii. the assignment

$$\begin{aligned} Q : C^\infty(M) &\longrightarrow \mathrm{End}\, \Gamma(L) \\ f &\longmapsto Q_f, \end{aligned}$$

where

$$Q_f = \nabla_{X_f} + \mathrm{i} c\, f.$$

The pair (L, ∇) called a prequantum line bundle on (M, ω), and the operator

$$Q_f = \nabla_{X_f} + \mathrm{i} c\, f$$

is said to be the quantum operator associated to $f \in C^\infty(M)$.

2. Polysymplectic manifolds

Definition

Fix a vector space V.

Definition

A *V-symplectic structure* on M is a closed, nondegenerate 2-form $\omega \in \Omega^2(M, V)$.

symplectic: $\omega \in \Omega^2(M)$

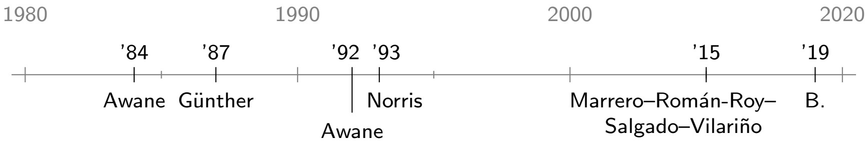
polysymplectic: $\Omega^2(M, V)$

multisymplectic: $\Omega^{k+1}(M)$

Definitions and statements of results **extend naturally**.

Many ways to extend definitions.

Background



Awane	k-symplectic geometry
Günther	polysymplectic geometry
Norris	n-symplectic geometry
Marrero–Román-Roy–Salgado–Vilariño	reduction
B.	V-symplectic geometry

Examples i

Example (Symplectic sums)

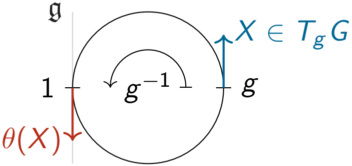
Suppose ω_1, ω_2 are symplectic forms on M and define $\omega = \omega_1 \oplus \omega_2$ by

$$\omega(X, Y) = \big(\omega_1(X, Y), \omega_2(X, Y)\big)$$

Then ω is an $\mathbb{R}^2$-symplectic form on M.

Example (Discrete-center Lie groups)

Fix a Lie group G with discrete center and let $\theta \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan form on G. Then $\omega = -\mathrm{d}\theta \in \Omega^2(G, \mathfrak{g})$ is a $\mathfrak{g}$-symplectic form on G.



$$\theta(X) = (\lambda_{g^{-1}})_* X$$

Examples ii

Example (Polyphase space)

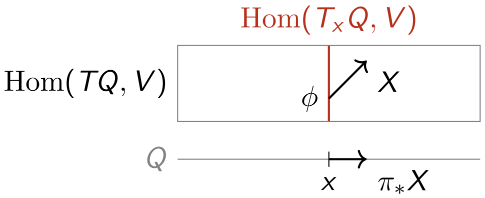
Let Q be a smooth manifold and consider the bundle

$$\pi : \mathrm{Hom}(TQ, V) \to Q$$

The *canonical* 1-*form* is

$$\theta_\phi(X) = \phi(\pi_* X), \qquad X \in T_\phi \mathrm{Hom}(TQ, V)$$

and the *canonical* V*-symplectic structure* is $\omega = -\mathrm{d}\theta$.



$$\begin{array}{c} \phi : T_x Q \longrightarrow V \\ \Cup \\ \pi_* X \end{array}$$

Polysymplectic Hamiltonian symmetries

Definition

The *Hamiltonian vector field* $X \in \mathfrak{X}(M)$ associated to a function $h \in C^\infty(M, V)$ is defined by

$$\mathrm{d}h = -\iota_X \omega$$

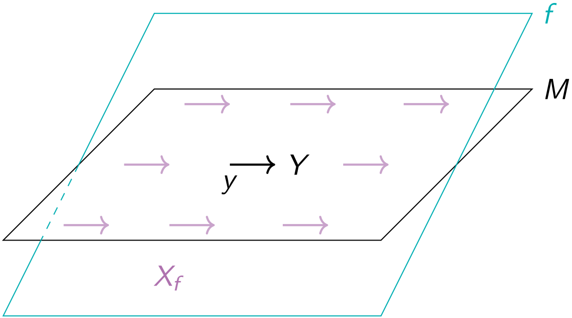
Conversely, h is called a *Hamiltonian function* for X.

1. Unlike the symplectic case, **not every function is Hamiltonian**.
2. $C^\infty_H(M, V)$ is a Lie algebra with *bracket* given by

$$\{f, h\} = X_f h$$

Transitivity

$$\forall Y \in TM \ : \exists f \in C^\infty_H(M,V) \ : \ Y = X_f(y)$$



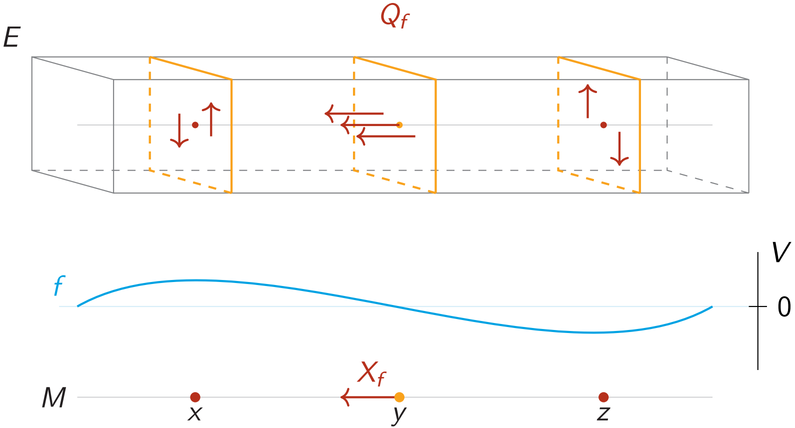
Definition

We say that (M,ω) is *transitive* if every tangent vector $Y \in TM$ extends to a Hamiltonian vector field $X_f \in \mathfrak{X}(M)$.

3. Polysymplectic prequantization

Prequantization – idea

Lift the Hamiltonian symmetries of (M, ω) to the space of sections of a Hermitian vector bundle $E \to M$.



$$Q : C^\infty_H(M, V) \longrightarrow \mathrm{End}\,\Gamma(E)$$

Prequantization – preliminary definition

Preliminary Definition

A *prequantization* of (M, ω) consists of a Hermitian vector bundle $E \to M$ and a faithful first-order* Lie algebra representation

$$Q : C^\infty_H(M, V) \to \mathrm{End}\,\Gamma(E) \qquad \text{Lie alg. hom. property}$$

preserving the inner product on the subspace of smooth L^2 sections of $E \to M$, such that

$$Q_f(s\psi) = (X_f s)\psi + sQ_f\psi \qquad \text{Hamiltonian lifting property}$$

for $f \in C^\infty_H(M, V)$, $s \in C^\infty(M)$, $\psi \in \Gamma(E)$.

Problem: What does it mean to be L^2? In the symplectic setting, there is a measure, ω^n, on (M^{2n}, ω), unique up to rescaling, that is preserved by the Hamiltonian dynamics.

* $(Q_f\psi)_x = 0$ whenever f vanishes to first order at x

Invariant measures

Definition

An *invariant measure* on (M^n, ω) is a volume form $\eta \in \Omega^n(M)$ that is preserved by every Hamiltonian vector field on M.

- Existence of a nonzero η is not guaranteed
- $\left.\begin{array}{l}\text{Existence of } \eta \neq 0 \\ \text{transitivity of } (M,\omega)\end{array}\right\} \implies$ uniqueness of η up to rescaling

Definition

An *algebra of (classical) observables* $\mathcal{O}$ is any Lie subalgebra of $C^\infty_H(M, V)$.

We define invariant measures with respect to an algebra of classical observables in the natural way.

*We will assume that $\mathcal{O}$ contains the constant functions.

Prequantization – motivating definition

Fix a V-symplectic manifold (M, ω), an algebra of classical observables $\mathcal{O} \subseteq C^\infty_H(M, V)$, and a nonzero $\mathcal{O}$-invariant measure η on M.

Definition

A prequantization of $(M, \omega, \mathcal{O}, \eta)$ consists of a Hermitian vector bundle $E \to M$ and a faithful first-order Lie algebra representation

$$Q : \mathcal{O} \to \mathrm{End}\,\Gamma(E) \qquad \text{Lie alg. hom. property}$$

preserving the inner product on the subspace of smooth L^2 sections of $E \to M$, such that

$$Q_f(s\psi) = (X_f s)\psi + sQ_f\psi \qquad \text{Hamiltonian lifting property}$$

for $f \in \mathcal{O}$, $s \in C^\infty(M)$, $\psi \in \Gamma(E)$.

For simplicity, we will assume $\mathcal{O} = C^\infty_H(M, V)$ and $\eta \neq 0$ exists.

The induced V-linear connection on $E \to M$

Consider the Lie subalgebra $V \subseteq \mathcal{O}$.

For all $v \in V$, the equality

$$Q_v(s\psi) = \underbrace{(X_v s)}_{0}\psi + sQ_v\psi = sQ_v\psi, \quad \forall s \in C^\infty(M), \psi \in \Gamma(E),$$

implies that $Q_v \in \mathrm{End}\,\Gamma(E)$ is tensorial.

We obtain an induced Lie algebra representation of V on the fibers of E,

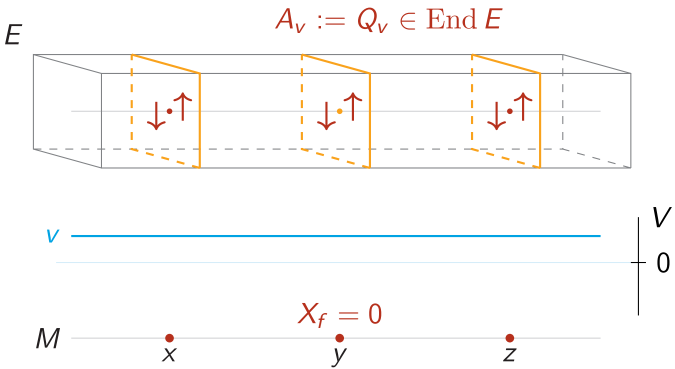
$$v \mapsto A_v \in \mathrm{End}\, E,$$

and $E \to M$ inherits the structure of a bundle of V-representations.

Proposition

If (M, ω) is transitive, then $\nabla_X \psi = (Q - A)_{f_X}\psi$ defines a V-linear covariant derivative on E.

Scalar multiplication and the induced connection



$A : V \to \mathrm{End}\, E$ scalar multiplication

$\nabla_{X_f} := (Q - A)_f$ V-linear covariant derivative on E

Prequantum vector bundles

Definition

A *prequantum vector bundle* (E, ∇, A) on (M, ω) consists of a

1. Hermitian vector bundle $E \to M$,
2. fiberwise V-module structure $A : V \to \mathrm{End}\, E$,
3. unitary connection ∇ on E with $\nabla A = 0$,

such that $F^\nabla = -A_\omega$, i.e. $F^\nabla(X_f, X_h) = -A_{\omega(X_f, X_h)}$ for all f, h.

Theorem

If (M, ω) is transitive and connected, then there is a natural correspondence:

$$\{\textit{prequantizations}\} \longleftrightarrow \{\textit{prequantum vector bundles}\}$$

$$\begin{aligned} Q : C^\infty_H(M, V) &\longrightarrow \mathrm{End}\, \Gamma(M, E) \\ f \quad &\longmapsto \nabla_{X_f} + A_f \end{aligned}$$

Some questions

1. What happens when (M, ω) is *not* transitive?
2. Does the Lie algebra homomorphism property imply the first-order condition?
3. What happens when we remove the first-order condition?
4. When does an invariant measure η on (M, ω) exist?
5. What other "nice" properties do transitive polysymplectic manifolds exhibit?

Thank you!