# Curvature & holonomy on double Lie groupoids

1. The Ambrose-Singer theorem
2. Lie groupoids & Lie algebroids
3. Double Lie groupoids & LA-groupoids
4. Higher Ambrose-Singer

Joint work in progress with Derek Krepski

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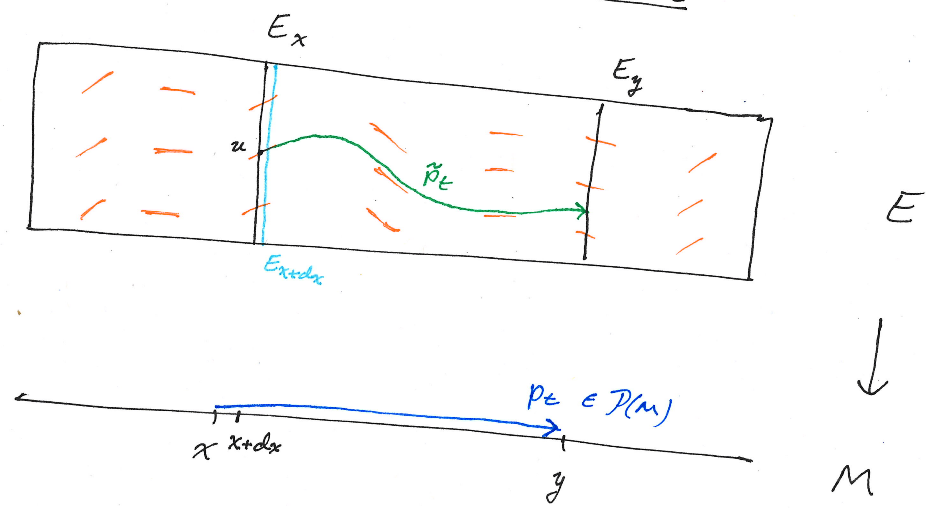
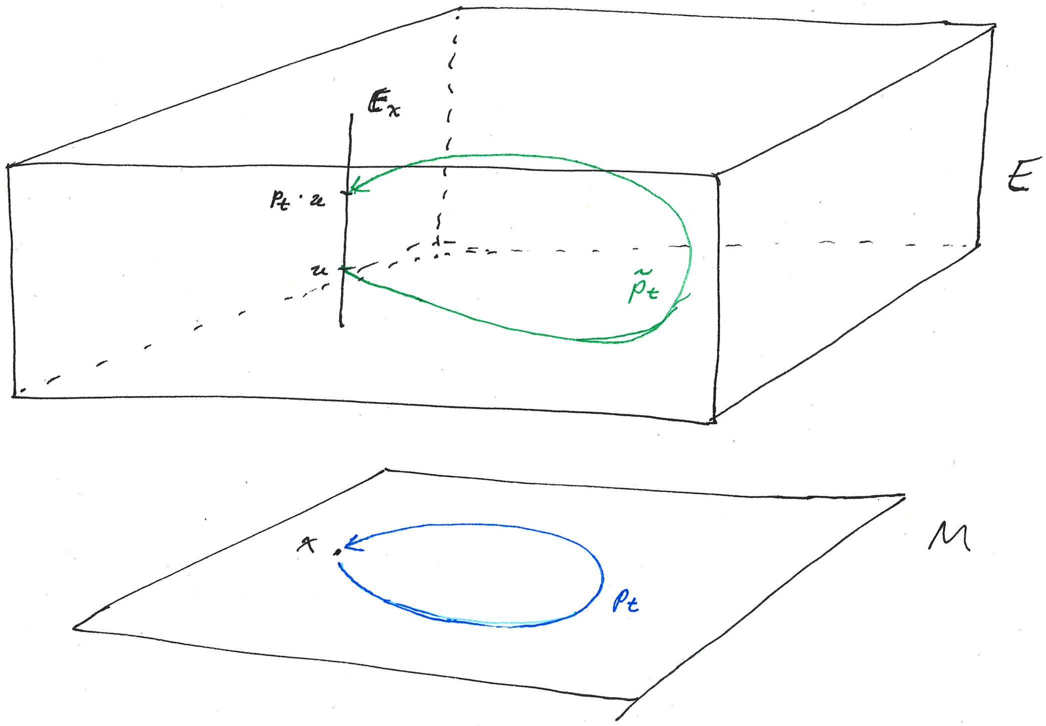
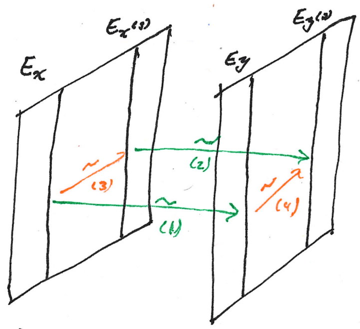
## 1. The Ambrose-Singer theorem



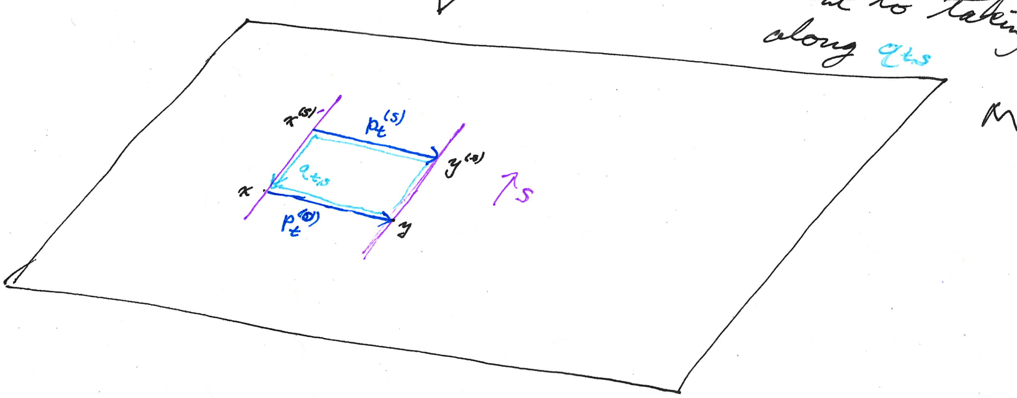

* connection: identification of infinitesimally close fibers
* parallel transport: the induced identification of fibers along paths $p_t \in \mathcal{P}(M)$.
* holonomy: the group $Hol_x \leq Aut(E_x)$ given by parallel transport along all loops $p_t$ based at $x \in M$.

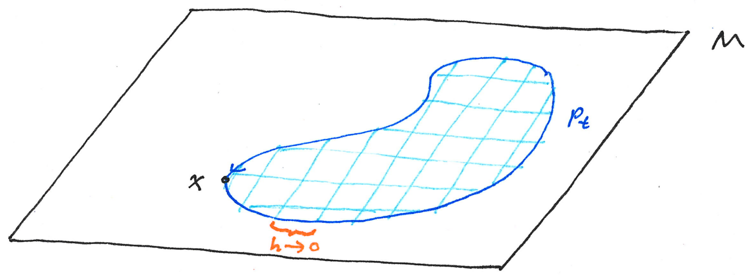



* <u>curvature</u>: the instantaneous rate of change of a connection with respect to itself.




To compare (1) & (2),
we need (3) & (4).
Equivalent to taking parallel transport along $q_{t,s}$

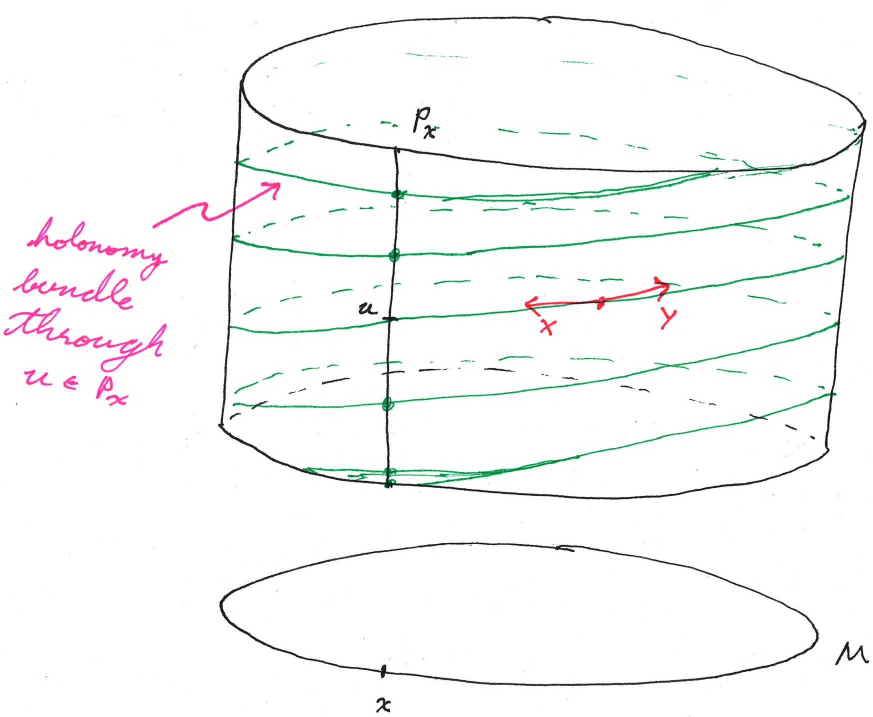





<u>idea</u>: $$\int_\Omega \underbrace{D\ \text{connection}}_{\text{curvature}} = \int_{p_t} \underbrace{\text{connection}}_{\text{holonomy}}$$

Ambrose-Singer, 1953:

$\boxed{\text{Curvature generates holonomy.}}$




connection

$\wr$

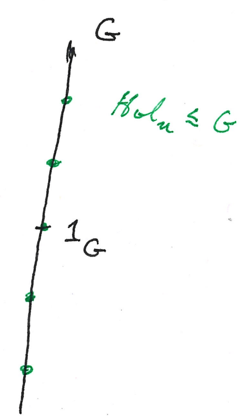
$\alpha \in \Omega^1(P, \mathfrak{g})$

$d\alpha \in \Omega^2(P, \mathfrak{g})$

$\wr$

curvature (on P)

$\boxed{\text{the } d\alpha(X,Y) \in \mathfrak{g} \text{ generate } Hol_u}$



## 2. Lie groupoids & Lie algebroids

Mackenzie, *General Theory of Lie Groupoids & Lie Algebroids*, (2005)

**def.** A *Lie groupoid* comprises

i. manifolds $M, G$

ii. smooth maps

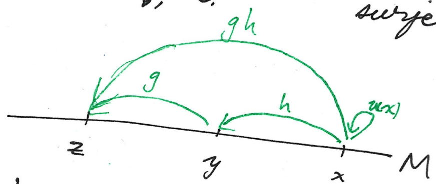
$$\begin{matrix} G \\ t \downarrow\downarrow s \\ M \end{matrix} \qquad \begin{matrix} G \\ \uparrow u \\ M \end{matrix} \qquad G \,{}_s\!\times_t G \longrightarrow G$$

groupoid ($G$), base ($M$)

subject to

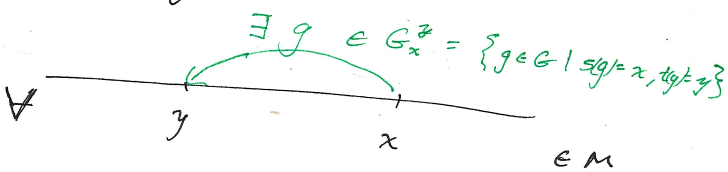
*algebraic conditions*
grouplike conds:
inverses, associativity, etc.

*smooth conditions*
$s,t : G \to M$
surjective submersions

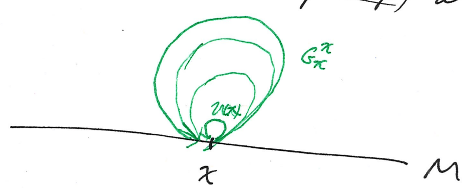



$\chi = (t,s) : G \to M$ is the *anchor map*.

$G \overset{\downarrow\downarrow}{M}$ is *transitive* if




The *adjoint group* (*vertex group*) at $x \in M$ is $\overset{(\mathcal{L}G)_x}{\overset{\shortparallel}{G_x^x}}$.




$A = \ker ds|_M$

$\rho = dt$

**def.** A *Lie algebroid* comprises

i. a vector bundle $\begin{matrix} A \\ \downarrow \\ M \end{matrix}$

ii. a Lie bracket

$$[\,,\,]_A : \Gamma A \times \Gamma A \longrightarrow \Gamma A$$

and bundle map

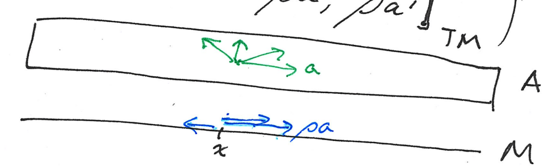
$$\begin{matrix} A \\ \rho \downarrow \\ TM \end{matrix}$$

called the *anchor*.

s.t.

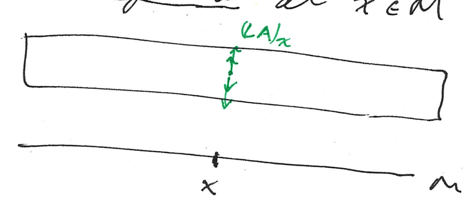
$$[a, \overset{C^\infty(M)}{f} a']_A = \underbrace{\rho(a) f}_{C^\infty(M)} \cdot a' + f\,[a,a']_A$$

$$\left( \Rightarrow \rho[a,a']_A = [\rho a, \rho a']_{TM} \right)$$



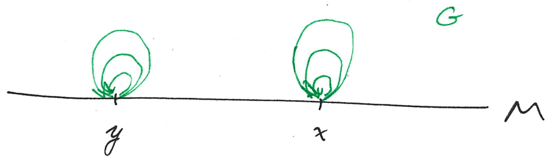

$\begin{matrix} A \\ \downarrow \\ M \end{matrix}$ is *transitive* if $\rho$ is surjective.

The *adjoint algebra* at $x \in M$ is $(\mathcal{L}A)_x = \ker \rho_x$.



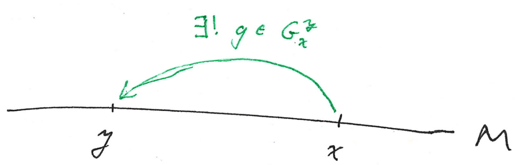
ex Lie gp $G$


* Lie gp bd



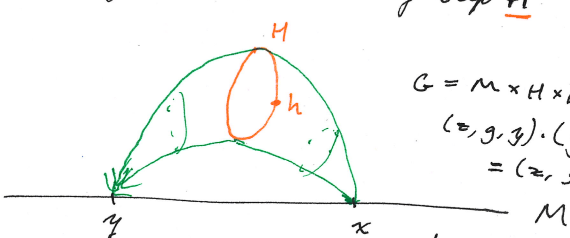

$LG = G$

* pair gpd




$LG = uM$

* trivial gpd w/ structure group $H$

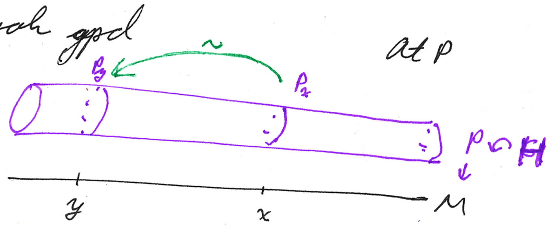



$G = M \times H \times M$

$(z, g, y) \cdot (y, h, x) = (z, gh, x)$

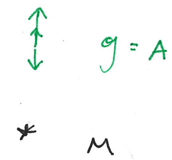
$LG = M \times H$

* Atiyah gpd at $P$




$L(\mathrm{at}P) = \mathrm{ad}P$

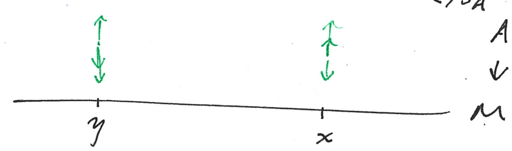
* Lie alg $\mathfrak{g}$




$[\,,]_A = [\,,]_{\mathfrak{g}}$

$\rho = 0$

* Lie alg bd

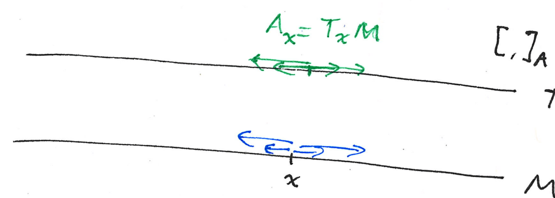



$[\,,]_A = [\,,]_{\mathfrak{g}}$

$\rho = 0$

$LA = A$

* $TM \to M$



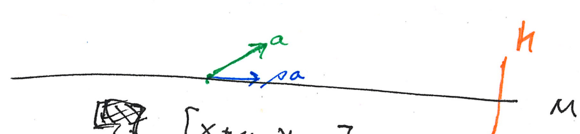

$[\,,]_A = [\,,]_{TM}$

$\rho = id$

$LA = 0 \to M$

* triv. algbd w/ str alg $\mathfrak{h}$

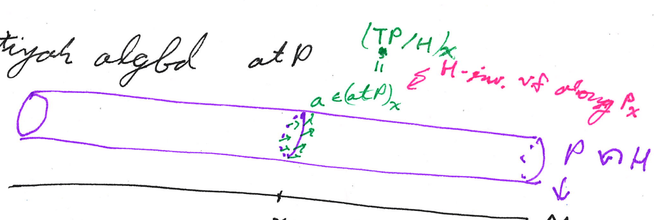
$A = TM \oplus (M \times \mathfrak{h})$




$[X+u, Y+v]_A = [X,Y]_{TM} + Xv - Yu + [u,v]_{\mathfrak{h}}$

$LA = M \times \mathfrak{h}$

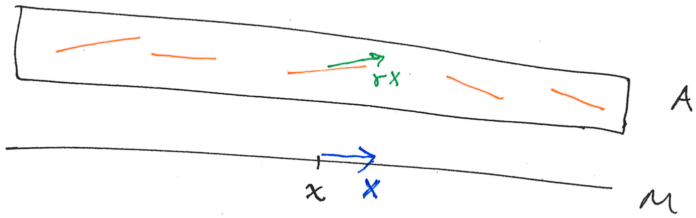
* Atiyah algbd at $P$




$L(\mathrm{at}P) = \mathrm{ad}P \hookrightarrow (TP_x)/H$

def. A Lie algebroid connection is a vector bundle lift of $\rho$. $(\Rightarrow A$ is transitive$)$

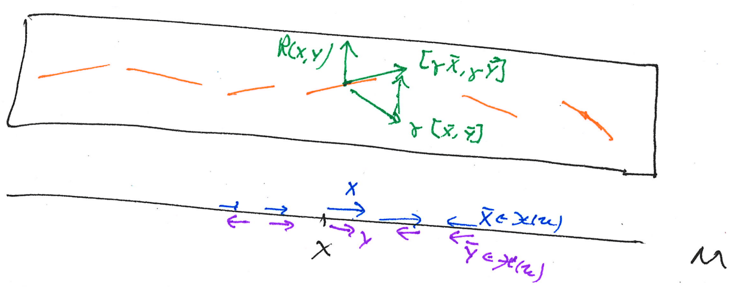
$$\begin{tikzcd} A \arrow[d, "\rho"] \\ TM \arrow[u, bend left, "\gamma"] \end{tikzcd}$$




The curvature of $\gamma$ is:

$$R^\gamma : TM \oplus TM \longrightarrow LA$$
$$(X, Y) \longmapsto [\gamma\bar{X}, \gamma\bar{Y}] - \gamma[\bar{X}, \bar{Y}]$$

vertical part of $[\gamma\bar{X}, \gamma\bar{Y}]$




fact. $\nabla^L_X a = [\gamma X, a]$ defines a connection on $\begin{matrix} LA \\ \downarrow \\ M \end{matrix}$.

def. Let $L^\gamma \subseteq L$ be the sub-LAB (Lie algebra bundle) generated by the image of $R^\gamma$ and preserved by $\nabla^L$.

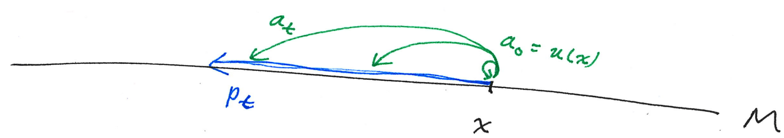
def The $\gamma$-curvature reduction of $\begin{matrix} A \\ \downarrow \\ M \end{matrix}$ is $\begin{matrix} A^\gamma = \gamma(TM) \oplus L^\gamma \\ \downarrow \\ M \end{matrix}$.

system of parallel transport　　　　　　paths

def. A path connection $\Gamma : \mathcal{P}(M) \longrightarrow \mathcal{P}(G)$ is a lift of paths on $M$ to paths on $G$, satisfying smooth and algebraic conditions

compositional

$\left(\Rightarrow \begin{smallmatrix} G \\ \downarrow\downarrow \\ M \end{smallmatrix} \text{ locally transitive}\right)$




def. The $\Gamma$-holonomy groupoid $Hol(\Gamma) \subseteq G$ is the image of $\Gamma$ in $G$.

* The holonomy group at $x \in M$ is the vertex group $Hol_x(\Gamma) \leq G_x$.

path connections on $\begin{smallmatrix} G \\ \downarrow\downarrow \\ M \end{smallmatrix}$ $\longleftrightarrow$ Lie algebroid connections on $\begin{smallmatrix} AG \\ \downarrow \\ M \end{smallmatrix}$

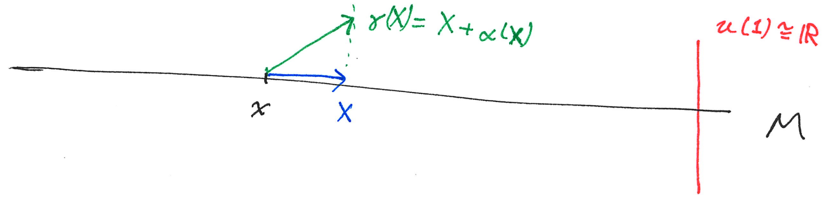
thm. (Mackenzie, '87)* $A\,Hol(\Gamma) = A^{\gamma}$.

generated by curvature

$\Longrightarrow$ Ambrose-Singer thm when $G = \mathrm{at}\,P$.

ex $G=$ triv. $U(1)$ gpd on $M$.

$$AG = TM \oplus \underbrace{\underline{u(1)}}_{\cong\, M\times u(1)}$$




$\alpha \in \Omega^1(M) \cong \Omega^1(M, u(1))$

$$\begin{aligned}
R(X,Y) &= [\gamma X, \gamma Y] - \gamma[X,Y] \\
&= [X + \alpha X, Y + \alpha Y] - \left([X,Y] + \alpha[X,Y]\right) \\
&= \cancel{[X,Y]} + X\alpha(Y) - Y\alpha(X) - \cancel{[X,Y]} + \alpha([X,Y]) \\
&= d\alpha(X,Y)
\end{aligned}$$

$\therefore \quad R = d\alpha$

$$\underset{\subseteq\, \underline{u(1)}}{L^\gamma} = \begin{cases} \underline{0} & \text{if } \alpha \text{ is closed} \\ \underline{u(1)} & \text{otherwise} \end{cases}$$

$$\underset{\subseteq\, \underline{TM\oplus u(1)}}{A^\gamma} = \begin{cases} \gamma(TM) & \text{———}\,//\,\text{———} \\ TM \oplus u(1) & \end{cases}$$

$$\underset{\subseteq\, G}{Hol(\Gamma)} = \begin{cases} \overset{\text{locally}}{\cong} M^2 \ \text{pair gpd} & \text{———}\,//\,\text{———} \\ G & \end{cases}$$

# 3. Double Lie groupoids & LA-groupoids

<u>def.</u> A <u>double Lie groupoid</u> comprises four Lie groupoids:

double groupoid

horizontal str.

vertical gpd

$$\begin{tikzcd} \boxed{S \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & V} \arrow[d, shift left, "\alpha"] \arrow[d, shift right, "\beta"'] \\ H \arrow[u, phantom] \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & M \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[d, shift right, "\beta"'] \arrow[d, shift left, "\alpha"] \\ H \end{tikzcd}$$

vertical str.

horizontal gpd

base

(we will assume all vertical structures are transitive throughout.)

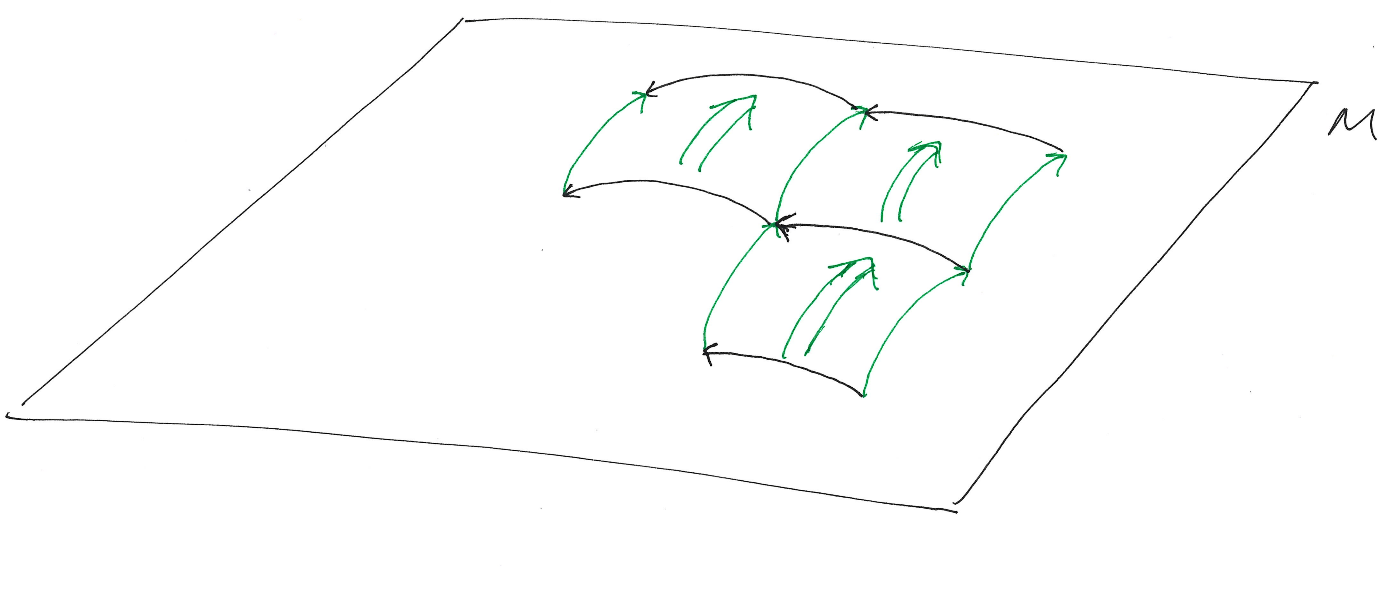
subject to

<u>algebraic conditions</u>

$s, t, u, m$ groupoid maps wrt $\alpha, \beta, \varepsilon, \nu$

<u>smooth conditions</u>

$\alpha s = s \alpha : S \longrightarrow M$
surjective submersion



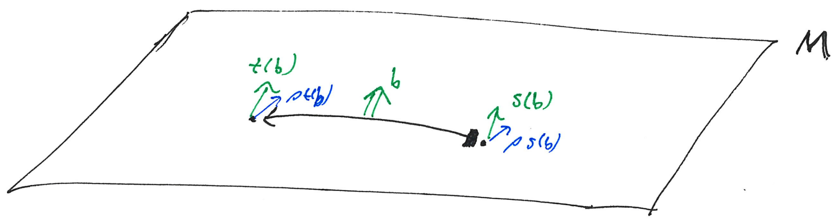
<u>def.</u> An LA-groupoid comprises

i. two vertical Lie algebroids $\begin{matrix} B \\ \downarrow \\ H \end{matrix}$, $\begin{matrix} A \\ \downarrow \\ M \end{matrix}$, with

ii. two horizontal groupoid structures

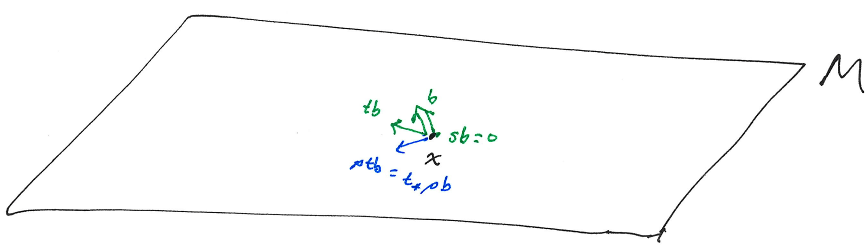
$$\begin{tikzcd} B \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[d] & A \arrow[d] \\ H \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & M \end{tikzcd}$$

s.t. all vertical (resp. horizontal) structure maps are horizontal (resp. vertical) homomorphisms.




The <u>core</u> bundle is

$$\begin{matrix} C = \ker s\big|_{M \subset H} \\ \downarrow \\ M \end{matrix}$$




$t : C \longrightarrow A$ Lie algebroid homomorphism def. An <u>LA-groupoid connection</u> is a compatible pair of Lie algebroid connections

$$\begin{tikzcd} B \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\rho"] & A \arrow[d, "\rho"] \\ TH \arrow[r, shift left] \arrow[r, shift right] \arrow[u, bend left, "\gamma"] & TM \arrow[u, bend left, "\gamma"] \end{tikzcd}$$

def. A <u>(double) path connection</u> is a pair of path connections

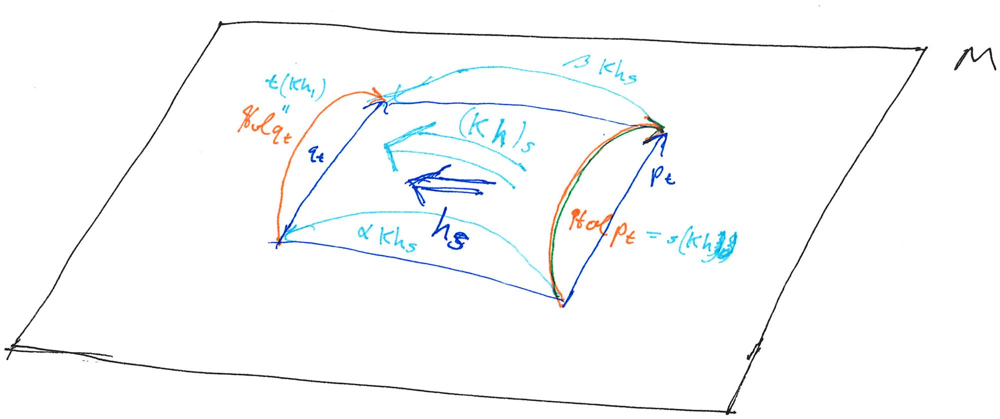
$$\begin{tikzcd} \mathcal{P}(S) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{P}(V) \\ \mathcal{P}(H) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "\Gamma"] & \mathcal{P}(M) \arrow[u, "\Gamma"'] \arrow[ul, "K"'] \end{tikzcd}$$

homotopies

def. A path curving $K : \mathcal{P}^2(M) \longrightarrow \mathcal{P}(S)$ is a path connection on

$$\begin{tikzcd} \mathcal{H}ol^* S \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{P}(M) \end{tikzcd}$$

$$\mathcal{H}ol : \mathcal{P}(M) \longrightarrow V$$
$$\underset{0 \le t \le t_1}{p_t} \longmapsto (\Gamma p_t)_{t_1}$$



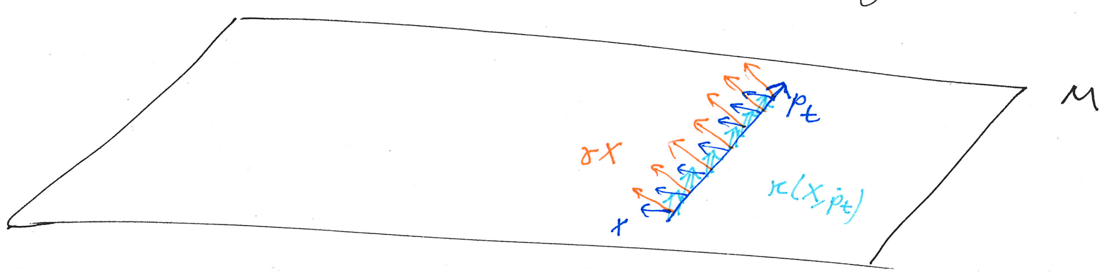
<u>def</u> An LA-groupoid curving $\kappa \in \Omega^2(M, LC)$ for $\gamma$ is a lift of $R^\gamma \in \Omega^2(M, LA)$ by $t : LC \to LA$

<u>idea:</u> $\kappa$ and $\gamma$ together induce a Lie algebroid connection $\bar{\kappa}$ on

$$\begin{array}{c} \mathcal{H}ol^* S \\ \downarrow\downarrow \\ \mathcal{P}(M) \end{array}$$

id 2-arrow of $\gamma X$

$\kappa(X, \dot{p}_t)$ adjusts $\alpha(\gamma X) = \gamma(d\alpha(X))$ along $p_t$




* ~~If~~ $R^\gamma = 0$ and $\kappa = 0$, then $K$ is determined by the resulting local trivialization.

* a path (resp. LA-groupoid) <u>connective structure</u> is a path (resp. LA groupoid) connection ~~$K(\kappa)$~~ $\Gamma(\gamma)$ and curving $K$ $(\kappa)$.

* The <u>3-curvature</u> of $(\kappa, \gamma)$ is

$\gamma$-induced connection on $C \subseteq B|_M$

$$d^\gamma \kappa = \nabla_X \kappa(Y,Z) - \nabla_Y \kappa(X,Z) + \nabla_Z \kappa(X,Y) - \kappa([X,Y],Z) + \kappa([X,Z],Y) - \kappa([Y,Z],X)$$
$$\in \ker t \subseteq C.$$

## 4. Higher Ambrose–Singer

ex. double Lie gpd:

U(1)-bundle

$$\begin{tikzcd} M \times U(1) \times M \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M^2 \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

U(1)-trivial gpd

pair gpd

unit gpd

LA-groupoid:

u(1)-bundle

$$\begin{tikzcd} TM \oplus \underline{u(1)} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \arrow[d] \\ M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

u(1)-trivial algebroid

connection:

$$\begin{tikzcd} \overset{B}{\overset{\shortparallel}{TM \oplus \underline{u(1)}}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\pi_*"] & \overset{A}{\overset{\shortparallel}{TM}} \arrow[d, "id"] \\ TM \arrow[r, shift left] \arrow[r, shift right] \arrow[u, bend left, "i"] & TM \arrow[u, bend left, "id"] \end{tikzcd}$$

$C = \ker t|_M \subseteq B|_M$

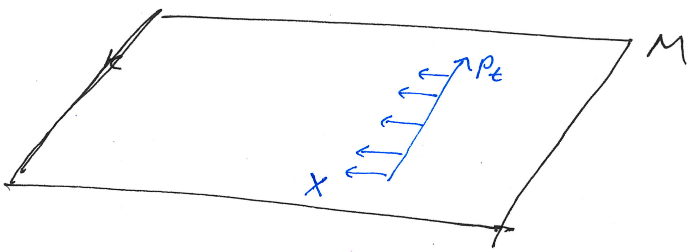
$$\begin{tikzcd} LC = \underline{u(1)} \arrow[rr, "0"] \arrow[dr] & & LA = \underline{0} \arrow[dl] \\ & M & \end{tikzcd}$$

$R^\sigma = 0.$

$\kappa \in \Omega^2(M, \underline{u(1)}) \cong \Omega^2(M).$

$(\kappa,\gamma)$-induced connection $\bar{\kappa}$ on $\begin{matrix} \mathcal{H}ol^* S \\ \downarrow\downarrow \\ M \end{matrix}$ : $\leftarrow$ $U(1)$-trivial gpd on paths $p_t, p'_t$ w/ equal endpoints: $\begin{cases} p^1_0 = p^2_0 \\ p^1_{t_f} = p^2_{t_f} \end{cases}$

$\Downarrow$



Note: In this example, admissible variations $X$ to $p_t$ must vanish at endpoints. (*) There are no arrows between paths w/ distinct endpoints.

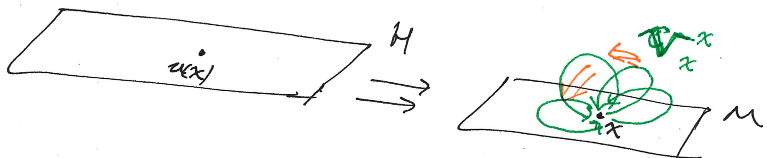
$$\bar{\kappa}(X) = \underset{\substack{\cap \\ T_{p_t}\mathcal{P}(M)}}{X} + \underbrace{\int_{p_t} \iota_X \kappa}_{\substack{\| \\ \alpha(X),\ \alpha \in \Omega^1(\mathcal{P}M, u(1))}}$$

$$\begin{aligned}
R^{\bar{\kappa}}(X,Y) &= [X + \alpha X, Y + \alpha Y] - \alpha[X,Y] \\
&= X\,\alpha(Y) - Y\,\alpha(X) - \alpha[X,Y] \\
&= X \int_{p_t} \iota_Y \kappa - Y \int_{p_t} \iota_X \kappa - \int_{p_t} \iota_{[X,Y]} \kappa \\
&= \int_{p_t} \mathcal{L}_X \iota_Y \kappa - \mathcal{L}_Y \iota_X \kappa - \iota_{[X,Y]} \kappa \\
&= \int_{p_t} \iota_Y \iota_X \underbrace{d\kappa}_{\substack{d^\nabla \kappa \\ \wr \\ \text{3-curvature}}} - d\,\iota_Y \iota_X \kappa \\
&= \underbrace{\kappa(X,Y)_{p_0} - \kappa(X,Y)_{p_{t_f}}}_{=0 \text{ by } (*)} + \int_{p_t} \iota_Y \iota_X d\kappa
\end{aligned}$$

$\therefore$ 3-curvature $d^\nabla \kappa$ generates $R^{\bar{\kappa}}$, which in turn generates 2-holonomy (i.e. $\kappa$-induced holonomy)

<u>Notes</u>

* It is not generally true that the boundary term $\kappa(x,y)\big|_{p_0}^{p_1}$ vanishes. However, it <u>is</u> true when we look at holonomy 2-groups, i.e. when we restrict to vertical 1-arrows based at $x \in M$ and the 2-arrows $D$ between them (with vertical sides $\alpha D = \beta D = u(x)$).




- The associated Lie 2-algebra to the vertex 2-group is represented by the differential crossed module

$$\begin{array}{c}(LC)_x\\ \downarrow\\ (LA)_x\end{array}$$

with action of $LA_x$ on $LC_x$ given by

$$a \cdot b = [ua, b]_B .$$

* If $R^\gamma = 0$, then $\kappa$ takes values in $\ker t \subseteq C$. The ~~[illegible]~~ $\gamma$-induced connection $\nabla$ on $C$ preserves $\ker t$, and the curvature $R\big|_{\ker t} = 0$. ~~Using the argument~~ This is sufficient to ~~[illegible]~~ adapt the argument of the previous page to show that the holonomy 2-groups are generated by $d^\nabla \kappa = d\kappa$ whenever $R^\gamma = 0$.