Existence of surface energy minimizing partitions of space satisfying volume constraints

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For many years, problems involving partitions of space $\mathbb{R}^n$ (for $n = 2, 3, \ldots$) have been of interest in mathematics, materials science, biology, image processing, and many other fields. It is natural to consider partitions of space into regions having specified volumes, as with materials of fixed volumes attempting to find a least-energy configuration (e.g., soap bubble clusters, immiscible fluids, polycrystals). Understanding, for instance, the possible singularities in energy minimizers would improve our insight into and ability to predict properties of polycrystalline materials.

The first rigorous proof of the existence and regularity of such surface energy minimizing partitions was given by Fred Almgren in 1976, in the general context of geometric measure theory. He considered a surface energy functional

$$SE = \sum_{i,j} \int_{\partial^* K_i \cap \partial^* K_j} \phi_{ij} dA,$$

where each ϕ_{ij} is of the form $c_{ij}\phi$, for a fixed smooth norm ϕ, so that each interface has the same ideal crystal shape, or Wulff shape. An important special case is when each ϕ is the Euclidean norm, as with soap bubble clusters. The structure of singularities for soap bubble clusters is by now well-known, but many singularities observed in actual materials are not of those types, since most materials are polycrystalline, and since the relation $\phi_{ij} = c_{ij}\phi$ does not typically hold in practice, since different materials give rise to interfaces with different physical properties.

It is important, therefore, to establish existence, regularity, and singularity structure results with independent norms ϕ_{ij}, to extend the theory so that it can be applied to polycrystalline materials.

In this talk, I will present my proof that surface energy minimizing partitions satisfying volume constraints exist, assuming the ϕ_{ij}'s are smooth norms satisfying BV-ellipticity, a condition necessary and sufficient for lower semicontinuity of surface energy. The difficult part is the construction of a bounded, surface energy minimizing sequence. Once such a sequence is constructed, a compactness theorem for polycrystals yields a limit, and lower semicontinuity ensures the limit is surface energy minimizing. We will work in the general context of the sets of finite perimeter of geometric measure theory, without making *a priori* assumptions on smoothness of interfaces or combinatorial structure of minimizers.