EMBEDDING QUADRATIC NUMBER FIELDS IN CYCLOTOMIC FIELDS

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ABSTRACT. If $m > 2$ is an integer, call the cyclotomic field $C_m = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/m}$. We see $C_m/\mathbb{Q}$ is Galois and we show $Gal(C_m/\mathbb{Q})) \approx \mathbb{Z}_m^*$, the unit group of the integers modulo m. Suppose $d \neq 0$, 1 is a square free integer, $K = \mathbb{Q}(\sqrt{d})$ and the "discrminant" $\Delta = d$ if $d \equiv 1 \pmod 4$, and $\Delta = 4d$ otherwise. We show $K \subseteq C_{|\Delta|}$. [This is a special case of the Kronecker-Weber theorem that every abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field. See Greenberg, Amer. Math. Monthly 81 (1974), 601–607.] We define the Kronecker symbol, which is really a character $\chi_\Delta(x)$ for $x \in \mathbb{Z}$ and $(x,m) = 1$, simply to be ± 1 depending on whether the automorphism $\zeta \mapsto \zeta^x$ fixes $\sqrt{d}$ or not. We'll define the p-adic numbers and if time permits, see that when the Legendre symbol is defined it agrees with the Kronecker symbol: If p is an odd prime, then $\chi_\Delta(p) = \left(\frac{\Delta}{p}\right)$. We should have enough time to have some fun with the p-adics, say by using Hensel's Lemma to show there are an infinite number of primes $p \equiv 1 \pmod m$ or to prove quadratic reciprocity. Knowledge of Galois Theory is all that will be needed to understand this talk.

Keywords: Cyclotomic field, Galois group, p-adics, Hensel's Lemma, Legendre symbol, Kronecker symbol

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