1^{st} Some observations about covering sieves

$(\mathcal{C}, J)$ Grothendieck site:

i) $\forall C$, $max(C) \in Cov_J(C)$
ii) if $h: D \to C$ and $S \in Cov_J(C)$, $h^*(S) \in Cov_J(D)$
iii) If $S \in Cov_J(C)$ and $R \rightarrowtail y(C)$ s.t. $\forall (h: D \to C) \in S$, $h^*R \in Cov_J(D) \Rightarrow R \in Cov_J(C)$.

Claim $\Rightarrow$

1) If $R \supseteq S$ and $S \in Cov_J(C) \Rightarrow R \in Cov_J(C)$

Pf Let $(h: D \to C) \in S$, $E \in \mathcal{C}_0$,

then
$$\begin{tikzcd} h^*S(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S(E) \arrow[d, hook] \\ Hom(E,D) \arrow[r, "h^*"] & Hom(E,C) \end{tikzcd}$$

$\Rightarrow h^*S(E) \cong \{g: E \to D \mid hg \in S(E)\}$

Since $h \in S \Rightarrow \forall g: E \to D$ $hg \in S$

$\Rightarrow h^*S(E) = Hom(E,D)$ $\forall E$

$\Rightarrow h^*S = max(D) \in Cov_J(D)$

But $R \supseteq S \Rightarrow h^*(R) \supseteq h^*S = max(D) \Rightarrow h^*(R) = max(D) \in Cov_J(D)$!

and since $h \in S$ was arbitrary $\Rightarrow h^*R \in Cov_J(D)$ $\forall h \in S \Rightarrow R \in Cov_J(C)$ by iii). $\square$

Claim iii) $\Rightarrow$ iii)' If $S \in Cov_J(C)$ and $\forall f: D_f \to C$ $\exists R_f \in Cov_J(D_f)$, then $Q := \{fg \mid (f: D_f \to C) \in S, g \in R_f\} \in Cov_J(C)$.

Pf $\forall (f: D_f \to C) \in S$ $f^*Q = \{g: E \to D_f \mid fg \in Q\} \supseteq R_f$

So $f^*Q \in Cov_J(D_f)$ $\forall f \in S \Rightarrow Q \in Cov_J(C)$. $\square$

Claim If $R, S \in Cov_J(C)$ then
$$\begin{tikzcd} R \cap S \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d] \\ R \arrow[r, tail] & y(C) \end{tikzcd}$$
has $R \cap S \in Cov_J(C)$.

Pf: $\forall (f: D \to C) \in R$, $f^*(R \cap S) = \{g \mid fg \in R \cap S\}$, but $fg \in R$ $\forall g$

$\therefore$ $f^*(R \cap S) = f^*(S) \in Cov_J(D)$ $\forall f \Rightarrow R \cap S \in Cov_J(C)$ $\square$.

The "Plus Construction"

Let $(\mathcal{C}, J)$ be a Grothendieck site.

Let $X \in Set^{\mathcal{C}^{op}}$. Define a new presheaf X^+ as follows:

$$X^+(C) = \{(R,\phi) \mid R \in Cov_J(C),\ \phi : R \to X\}/\sim$$

$$(R,\phi) \sim (S,\psi) \iff \exists \underset{T \text{ a cov sieve.}}{T \subseteq R \cap S} \text{ s.t. } \phi_{|R\cap S} = \psi_{|R\cap S},$$

where $R \cap S = R \times_{y(C)} S$.

More categorically:

Let $Cov_J(C) \subseteq Sub(y(C))$ - poset, and consider the canonical functor

$$i_C : Cov_J(C) \longrightarrow Set^{\mathcal{C}^{op}}.$$

Consider the composite

$$(Cov_J(C))^{op} \xrightarrow{i_C^{op}} (Set^{\mathcal{C}^{op}})^{op} \xrightarrow{y(X)} Set$$

$X^+(C) = \varinjlim y(X) \circ i_C^{op}$, or slightly informally: $X^+(C) = \varinjlim_{S \in Cov_J(C)} Hom(S, X)$.

Need to show that X^+ is actually a presheaf:

Let $D \xrightarrow{f} C \rightsquigarrow f^* : Cov_J(C) \longrightarrow Cov_J(D)$

$$\begin{tikzcd} f^*R \arrow[r] \arrow[d] & R \arrow[d] \\ D \arrow[r, "f"] & C \end{tikzcd}$$

these assemble into a natural transformation $\alpha_f : i_D \circ f^* \Longrightarrow i_C$.

$\rightsquigarrow \forall R \in Cov_J(C)$ morphisms $Hom(R, X) \xrightarrow{y(X)(\alpha_f(R))} Hom(f^*R, X)$.

Let $\lambda^D : y(X) \circ i_D^{op} \Longrightarrow \Delta_{X^+(D)}$ be the colimiting cocone,

$$\lambda^D(S) : Hom(S, X) \longrightarrow X^+(D)$$
$$\phi : S \to X \longmapsto [(S, \phi)]$$

then $\exists\, \lambda_f : y(X)\circ i_C^{op} \Longrightarrow \Delta_{X^+(D)}$

$$\lambda_f(R) : \mathrm{Hom}(R,X) \xrightarrow{y(X)(\alpha_f(R))} \mathrm{Hom}(f^*R, X) \xrightarrow{\lambda^D_{f^*R}} X^+(D),$$

which is natural since α_f is.

So $\exists!\ X^+(f) : X^+(C) \longrightarrow X^+(D)$ s.t.

$$\begin{tikzcd} y(X)\circ i_C^{op} \arrow[r, Rightarrow, "\lambda_f"] \arrow[d, Rightarrow, "\lambda^C"'] \arrow[dr, phantom, "\circlearrowleft"] & \Delta_{X^+(D)} \\ \Delta_{X^+(C)} \arrow[ur, Rightarrow, "\Delta_{X^+(f)}"'] & \end{tikzcd}$$

Concretely: $X^+(f)([R,\psi]) = [f^*R,\ \psi\circ\alpha_f(R)]$.

<u>Claim</u>: $X \longmapsto X^+$ is a functor $Set^{\mathcal{C}^{op}} \longrightarrow Set^{\mathcal{C}^{op}}$.

Suppose $\theta : X \longrightarrow F$ in $Set^{\mathcal{C}^{op}}$, then $\forall\, C \in \mathcal{C}_0$ we

have $\quad y(X)\circ i_C^{op} \xRightarrow{y(\theta)\circ id} y(F)\circ i_C^{op}$

$$\rightsquigarrow$$

$$X^+(C) = \varinjlim y(X)\circ i_C^{op} \xrightarrow{\theta^+(C)} \varinjlim y(F)\circ i_C^{op} = F^+(C)$$

$$[R,\phi] \longmapsto [R, \theta\phi]$$

which is easily seen to define a natural transformation

$\theta^+ : X^+ \Longrightarrow F^+$ as desired.

<u>Note</u> There is also a canonical natural transformation

$\sigma : id_{Set^{\mathcal{C}^{op}}} \Longrightarrow (\cdot)^+$

$$\begin{tikzcd} X(C) \arrow[r, "\sim"] \arrow[rr, bend left, "\sigma_X(C) :="] & \mathrm{Hom}(y(C), X) = \mathrm{Hom}(\max(C), X) \arrow[r, "\lambda^C_{\max(C)}"] & X^+(C) \end{tikzcd}$$

$$x \longmapsto (\tilde{x} : y(C) \longrightarrow X) \longmapsto [(\max(C), \tilde{x})],$$

is natural in C. (follows from naturality of Yoneda), and

$$\sigma_X : X \longrightarrow X^+ \quad \text{is natural in } X.$$

<u>Prop</u> $(\cdot)^+ : \mathrm{Set}^{\mathcal{C}^{op}} \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$ preserves finite limits.

<u>Pf</u>: $\forall\, C$, $(\mathrm{Cov}_J(C))^{op}$ is a filtered poset, and filtered colimits commute with finite limits in Set, here:

If $F = \varprojlim F_i$ is a finite limit, $\forall\, C \in \mathcal{C}_0$

$$F^+(C) = \varinjlim_{R \in \mathrm{Cov}_J(C)} \mathrm{Hom}(R, F) = \varinjlim_{R \in \mathrm{Cov}_J(C)} \mathrm{Hom}(R, \varprojlim_i F_i)$$

$$\cong \varinjlim_{R \in \mathrm{Cov}_J(C)} \varprojlim_i \mathrm{Hom}(R, F_i) \underset{\text{filtered}}{\cong} \varprojlim_i \Big(\varinjlim_{R \in \mathrm{Cov}_J(C)} \mathrm{Hom}(R, F_i) \Big)$$

$$\cong \varprojlim_i (F_i)^+(C).$$

<u>Lemma</u>: Let $X \in \mathrm{Set}^{\mathcal{C}^{op}}$ and $F \in \mathrm{Sh}_J(\mathcal{C})$, and $g : X \longrightarrow F$.

Then $\exists!\ \tilde{g}$ s.t.
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\sigma_X"'] \arrow[dr, phantom, "\circlearrowleft"] & F \\ X^+ \arrow[ur, dashed, "\tilde{g}"'] & \end{tikzcd}$$

<u>Pf</u> Let $[(S, \varphi)] \in X^+(C)$,
$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d] & X \\ y(C) & \end{tikzcd} \rightsquigarrow \begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d] & X \arrow[r, "g"] \arrow[d, phantom, "\circlearrowleft"] & F \\ y(C) \arrow[urr, dashed, "\exists!\ \tilde{g}_C(S,\varphi)"'] & & \end{tikzcd},$$

since S is a covering sieve and F is a sheaf.

Note $\tilde{g}_C(S,\varphi)$ is well defined:

Suppose $(S,\varphi)\sim(R,\psi) \Rightarrow \exists\, \underset{\text{cov sieve.}}{T}\subset R\cap S$ s.t. $\varphi|_{R\cap S} = \psi|_{R\cap S}$.

But $T\in Cov_J(C)$, so

$$\begin{tikzcd}
T \arrow[r, tail] \arrow[d, tail] & S \arrow[d, tail] \arrow[ddr, bend left, "\varphi"] & \\
R \arrow[r, tail] \arrow[drr, bend right, "\psi"'] & y(C) & \\
 & & X \arrow[r, "g"] & F
\end{tikzcd}$$

we have that $\tilde{g}_C(S,\varphi)$ and $\tilde{g}_C(R,\psi)$ restrict to the same map $T\longrightarrow F$, and hence are equal, since T is a cov. sieve and F a sheaf. (just need F separated)

A similar argument shows that $\tilde{g}_C : X^+(C)\longrightarrow Hom(y(C),F)$ is natural in C.

$$\rightsquigarrow \quad X^+ \overset{\tilde{g}}{\Longrightarrow} Hom(y(\cdot),F)\cong F.$$

Lemma For any $X\in Set^{C^{op}}$, X^+ is separated.

Let $T\in Cov_J(C)$ and consider the map

$$X^+(C)\cong Hom(y(C),X^+)\overset{r}{\longrightarrow} Hom(T,X^+).$$

We want to show that it is mono. Let $\underset{x}{\overset{\shortparallel}{[(S,\varphi)]}}, \underset{y}{\overset{\shortparallel}{[(R,\psi)]}}\in X^+(C)$ have the same image under r,

$x \leftrightsquigarrow \bar{x}: y(C)\longrightarrow X^+$, let $(h:D\to C)\in T(D)$:

$$\begin{tikzcd}
T(C) \arrow[r, tail] \arrow[d, "T(h)"'] & y(C)(C) \ni id_C \arrow[r, "\bar{x}_C"] \arrow[d, "y(C)(h)"'] \arrow[dr, phantom, "\circlearrowleft"] & X^+(C) \arrow[d, "X^+(h)"] \\
h\in T(D) \arrow[r, tail] & y(C)(D) \arrow[r, "\bar{x}_D"] & X^+(D)
\end{tikzcd}$$

$r(x): T\Longrightarrow X^+$, $r(x)(h) = X^+(h)(x) = [h^*S, \varphi\circ\alpha_h(S)]$

So $r(x) = r(y) \Rightarrow \forall \ h \in T(D) \ (\forall D)$

$$[[h^*S, \varphi\circ\alpha_h(S)]] = [(h^*R, \psi\circ\alpha_h(R))] \in X^+(D)$$

$\Rightarrow \forall h \ \exists \ T_h \in Cov_J(D)$ with $T_h \subset (h^*S \cap h^*R)$ s.t.

$$\varphi\circ\alpha_h(S)|_{T_h} = \psi\circ\alpha_h(R)|_{T_h}. \quad (*)$$

Let $U \rightarrowtail y(C)$ be the sieve

$$U = \{hg : g \in T_h, h \in T(D)\} \in Cov_J(C) \text{ and } U \subset R \cap S$$

This is a sieve since if $hg : E \to C$, $l : E' \to E$

$(hg)l = h(\underbrace{gl}_{T_h}) \in U$. (We already showed it's a covering sieve on page 0)

Note: If $h : D \to C$ and $S \rightarrowtail y(C)$

the p.b.

$$\begin{tikzcd} h^*S \arrow[r, "\alpha_h(S)"] \arrow[d, tail] & S \arrow[d, tail] \\ y(D) \arrow[r, "y(h)"] & y(C) \end{tikzcd}$$

is computed object-wise: $\forall D'$:

$$\begin{tikzcd} h^*S(D') \arrow[r, "\alpha_h(S)_{D'}"] \arrow[d, tail] & S(D') \arrow[d, tail] \\ y(D)(D') \arrow[r] & y(C)(D') \end{tikzcd}$$

$K \longmapsto^{\alpha_h(S)_{D'}} hK$

$$\begin{tikzcd} h^*S(D') \cong \{K : D' \to D \mid hK \in S(D')\} \arrow[r] \arrow[d] & S(D') \arrow[d] \\ y(D)(D') \arrow[r] & y(C)(D') \end{tikzcd}$$

$K \mapsto K$

$\Rightarrow$ if $T_h(D') \subseteq h^*S(D') \cap h^*R(D') \Rightarrow \forall g \in T_h(D')$
$hg \in S(D') \cap R(D')$.

also $\forall g \in T_h(D')$
$\varphi\circ\alpha_h(S)_{D'}(g) = \varphi(hg)$
$(*) \quad \shortparallel$
$\psi\circ\alpha_h(R)_{D'}(g) = \psi(hg)$

$\Rightarrow \varphi_{|U} = \psi_{|U} \Rightarrow [R, \varphi] = [(S, \psi)]$ ☒.

<u>Lemma</u> If X is separated, X^+ is a sheaf.

<u>Pf</u> Let $R \in Cov_J(C)$ and $\phi : R \longrightarrow X^+$.

Let $(f : C' \to C) \in R(C') \xrightarrow{\phi(C')} X^+(C')$,

$$\phi(C') = [(R_f, \phi_f)], \quad R_f \in Cov_J(C'), \quad \phi_f : R_f \longrightarrow X.$$

Let $S \in Cov_J(C)$ be defined by

$$S = \{fg \mid f \in R, g \in R_f\} \subset R.$$

Define $\psi : S \longrightarrow X$ by

$$\begin{aligned} \psi_{C''} : S(C'') &\longrightarrow X(C'') \\ fg &\longmapsto (\phi_f)_{C''}(g). \end{aligned}$$

$$\underset{R_f}{\underset{\Psi}{C''}} \xrightarrow{g} \underset{R}{\underset{\Psi}{C'}} \xrightarrow{f} C$$

It is not clear yet that ψ is natural or even well defined. We will show this is a minute. For now, assume this has been shown.

Note that is $r : Hom(y(C), X^+) \longrightarrow Hom(R, X^+)$ is the restriction map, then $r([S, \psi]) : R \longrightarrow X^+$, by the previous proof is of the form

$$\begin{aligned} r([S,\psi])_{C'} : R(C') &\longrightarrow X^+(C') \\ \underset{f}{\Psi} &\longmapsto [f^*S, \psi \circ \alpha_f(S)], \text{ and} \end{aligned}$$

$$(f^*S)(C'') = \{K : C'' \to C' \mid fK \in S\} \xrightarrow{\alpha_f(S)} S(C'')$$

$$\begin{tikzcd} K \arrow[r, mapsto] \arrow[rd, mapsto] \arrow[rd, phantom, "\star", near end, swap] & fK \arrow[d, "\psi_{C''}"] \\ (\phi_f)_{C''}(K). & X(C'') \end{tikzcd}$$

Note: if $(K : C'' \to C') \in R_f(C') \Rightarrow fK \in S$ (by defin)

$\Rightarrow R_f \subset f^*S \;\; \forall f$ and $\star \Rightarrow (\psi_{C''} \circ \alpha_f(S))|_{R_f} = \phi_f$.

$\therefore \quad r([S,\psi])_{C'}(f) = [(R_f, \phi_f)] = \phi_{C'}(f) \;\; \forall f \Rightarrow r([S,\psi]) = \phi.$

Since ϕ was arbitrary $\Rightarrow r$ is surjective, and X^+ is separated $\Rightarrow r$ is mono, $\therefore r$ is an isomorphism, and X^+ is a sheaf.

It now suffices to show that ψ is well-defined & natural.

<u>Well-defined:</u>

Suppose $f, f' \in R$, $g \in R_f$, $g' \in R_{f'}$ s.t.

$$\star \qquad \begin{tikzcd} & C' \arrow[dr, "f"] & \\ C'' \arrow[ur, "g"] \arrow[dr, "g'"'] \arrow[rr, phantom, "\circlearrowright"] & & C. \\ & D \arrow[ur, "f'"'] & \end{tikzcd} \qquad fg = f'g' \in R(C'')$$

WTS $(\phi_f)_{C''}(g) = (\phi_{f'})_{C''}(g')$.

Naturality for $\phi \Rightarrow$

$$\begin{tikzcd} f \arrow[r, mapsto] & R(C') \arrow[r, "\phi_{C'}"] \arrow[d, "R(g)"'] & X^+(C) \arrow[d, "X^+(g)"] \\ fg & R(C'') \arrow[r, "\phi_{C''}"] & X^+(C'') \end{tikzcd} \qquad f \mapsto [(R_f, \phi_f)]$$

so $\phi_{C''}(fg) = X^+(g)(\phi_{C'}(f)) = X^+(g)([R_f, \phi_f])$

$$= [g^* R_f, \phi_f \circ \alpha_g(R_f)]$$

$\star \Rightarrow [g^* R_f, \phi_f \circ \alpha_g(R_f)] = [g'^* R_{f'}, \phi_{f'} \circ \alpha_{g'}(R_{f'})] \Rightarrow$

$\exists \; T \in Cov_J(C'')$, $T \subseteq g^* R_f \cap g'^* R_{f'}$ s.t.

$$\phi_f \circ \alpha_g(R_f)|_T = \phi_{f'} \circ \alpha_{g'}(R_{f'})|_T \quad (**)$$

But note: $(g^* R_f)(E) = \{ \gamma : E \to C'' \mid g\gamma \in R_f \} \xrightarrow{\alpha_g(R_f)_E} R_f(E) \xrightarrow{(\phi_f)(E)} X(E)$

$\gamma \longmapsto g\gamma$

so $(**) \Rightarrow \forall \; \gamma \in T(E) \quad (\phi_f)(g\gamma) = (\phi_{f'})(g'\gamma)$.

Consider the naturality square

$$\begin{tikzcd} g \in R_f(C'') \arrow[r, "(\phi_f)_{C''}"] \arrow[d, "R_f(\gamma)"'] \arrow[dr, phantom, "\circlearrowleft"] & X(C'') \arrow[d, "X(\gamma)"] \\ g\gamma \in R_f(E) \arrow[r, "(\phi_f)_E"] & X(E) \end{tikzcd}$$

$\Rightarrow (\phi_f)(g\gamma) = X(\gamma)((\phi_f)_{C''}(g))$

hence: $\forall \gamma \in T$, $X(\gamma)((\phi_f)_{C''}(g)) = X(\gamma)(\phi_{f'})_{C''}(g')$. $(***)$

Note: Since X is separated, the canonical map

$$\begin{aligned} r_T : X(C'') &\longrightarrow \mathrm{Hom}(T, X) \\ z &\longmapsto \beta(z) : T \Longrightarrow X \end{aligned} \qquad (\gamma : E \to C'')$$

$$\begin{aligned} \beta(z)(E) : T(E) &\longrightarrow X(E) \\ \gamma &\longmapsto X(\gamma)(z). \end{aligned}$$

is mono.

So, by $(***) \Rightarrow (\phi_f)_{C''}(g) = (\phi_{f'})_{C''}(g')$.

Finally, need to show that $\begin{aligned} \psi_{C''} : S(C'') &\longrightarrow X(C'') \\ fg &\longmapsto (\phi_f)_{C''}(g) \end{aligned}$ is natural.

Suppose $h : C''' \to C''$, WTS

$$\begin{tikzcd} S(C'') \arrow[r, "\psi_{C''}"] \arrow[d, "S(h)"'] \arrow[dr, phantom, "\circlearrowleft"] & X(C'') \arrow[d, "X(h)"] \\ S(C''') \arrow[r, "\psi_{C'''}"] & X(C''') \end{tikzcd}$$

WTS: $\psi_{C'''}(f\overbrace{gh}^{\in R_f}) = X(h)(\psi_{C''}(fg))$

$\quad \Downarrow\!\!\shortparallel$

$(\phi_f)_{C'''}(gh)$

Note $\phi_f : R_f \to X$ is natural:

$$\begin{tikzcd} g \in R_f(C'') \arrow[r, "(\phi_f)_{C''}"] \arrow[d, "R_f(h)"'] \arrow[dr, phantom, "\circlearrowleft"] & X(C'') \arrow[d, "X(h)"] \\ gh \in R_f(C''') \arrow[r, "(\phi_f)_{C'''}"] & X(C''') \end{tikzcd}$$

$\boxed{2}$

<u>Cor</u> The inclusion $i: Sh_J(\mathcal{C}) \hookrightarrow Set^{\mathcal{C}^{op}}$ admits a left exact left adjoint $a \dashv i$.

<u>Pf</u> If X is arbitrary, X^+ is separated $\Rightarrow$ X^{++} is always a sheaf. By restriction of codomain,
$a := (\cdot)^+ \circ (\cdot)^+ : Set^{\mathcal{C}^{op}} \longrightarrow Sh_J(\mathcal{C})$.

There is a canonical natural transformation
$\eta : id \Longrightarrow a$ given by $\eta_X = (X \xrightarrow{\sigma_X} X^+ \xrightarrow{\sigma_{X^+}} X^{++} = aX)$.

Now, if $F \in Sh_J(\mathcal{C})$ and $g : X \longrightarrow iF$

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\sigma_X"'] \arrow[dd, bend right=60, "\eta_X"'] & iF \\
X^+ \arrow[d, "\sigma_{X^+}"'] \arrow[ur, dashed, "\exists!\, \tilde{g}"'] & \\
X^{++} = iaX \arrow[uur, dotted, "\exists!\, \tilde{\tilde{g}}"'] &
\end{tikzcd}$$

$\Rightarrow$ composition with η_X induces a bijection
$Hom(aX, F) \cong Hom(X, iF)$

$\Rightarrow$ η is the unit of an adjunction
$a \dashv i$.

Finally, since each $(\cdot)^+$ preserves finite limits, so does a.