

1st Some observations about covering sieves

$(\mathcal{C}, \mathcal{J})$ Grothendieck site.

- i) $\forall C, \max(C) \in \text{Cov}_{\mathcal{J}}(C)$
- ii) if $h: D \rightarrow C$ and $S \in \text{Cov}_{\mathcal{J}}(C)$, $h^*(S) \in \text{Cov}_{\mathcal{J}}(D)$
- iii) If $S \in \text{Cov}_{\mathcal{J}}(C)$ and $R \rightrightarrows y(C)$ s.t. $\forall (h: D \rightarrow C) \in S$, $h^*R \in \text{Cov}_{\mathcal{J}}(D) \Rightarrow R \in \text{Cov}_{\mathcal{J}}(C)$.

Claim \Rightarrow

1) IF $R \supseteq S$ and $S \in \text{Cov}_{\mathcal{J}}(C) \Rightarrow R \in \text{Cov}_{\mathcal{J}}(C)$

Pf Let $(h: D \rightarrow C) \in S$, $E \in \mathcal{C}_0$,

$$\begin{array}{ccc}
 \text{then } h^*S(E) \xrightarrow{h} S(E) & \Rightarrow & h^*S(E) \cong \{g: E \rightarrow D \mid hg \in S(E)\} \\
 \downarrow & & \downarrow \\
 \text{Hom}(E, D) \xrightarrow{h^*} \text{Hom}(E, C) & & \text{Since } h \in S \Rightarrow \forall g: E \rightarrow D \quad hg \in S \\
 & & \Rightarrow h^*S(E) = \text{Hom}(E, D) \quad \forall E \\
 & & \Rightarrow h^*S = \max(D) \in \text{Cov}_{\mathcal{J}}(D)
 \end{array}$$

But $R \supseteq S \Rightarrow h^*(R) \supseteq h^*S = \max(D) \Rightarrow h^*(R) = \max(D) \in \text{Cov}_{\mathcal{J}}(D)$

and since $h \in S$ was arbitrary $\Rightarrow h^*R \in \text{Cov}_{\mathcal{J}}(D) \quad \forall h \in S \Rightarrow R \in \text{Cov}_{\mathcal{J}}(C)$ by iii). \square

Claim iii) \Rightarrow iii)' If $S \in \text{Cov}_{\mathcal{J}}(C)$ and $\forall f: D_f \rightarrow C$

$\exists R_f \in \text{Cov}_{\mathcal{J}}(D_f)$, then $Q := \{fg \mid (f: D_f \rightarrow C) \in S, g \in R_f\} \in \text{Cov}_{\mathcal{J}}(C)$,

Pf $\forall (f: D_f \rightarrow C) \in S \quad f^*Q = \{g: E \rightarrow D_f \mid fg \in Q\} \supseteq R_f$

So $f^*Q \in \text{Cov}_{\mathcal{J}}(D_f) \quad \forall f \in S \Rightarrow Q \in \text{Cov}_{\mathcal{J}}(C)$. \square

Claim IF $R, S \in \text{Cov}_{\mathcal{J}}(C)$ then $R \circ S \rightarrow S$ has $R \circ S \in \text{Cov}_{\mathcal{J}}(C)$.

$$\begin{array}{ccc}
 R \circ S & \rightarrow & S \\
 \downarrow & & \downarrow \\
 R & \rightarrow & y(C)
 \end{array}$$

Pf: $\forall (f: D \rightarrow C) \in R, f^*(R \circ S) = \{g \mid fg \in R \circ S\}$, but $fg \in R \quad \forall g$

$\therefore f^*(R \circ S) = f^*(S) \in \text{Cov}_{\mathcal{J}}(D) \quad \forall f \Rightarrow R \circ S \in \text{Cov}_{\mathcal{J}}(C)$. \square

The "Plus Construction"

Let $(\mathcal{C}, \mathcal{J})$ be a Grothendieck site.

Let $X \in \text{Set}^{\text{cop}}$. Define a new presheaf X^+ as follows:

$$X^+(C) = \{ (R, \phi) \mid R \in \text{Cov}_{\mathcal{J}}(C), \phi: R \rightarrow X \} / \sim$$

$$(R, \phi) \sim (S, \psi) \iff \exists \begin{matrix} T \subseteq R \cap S \\ T \text{ a cover sieve.} \end{matrix} \text{ s.t. } \phi|_{T \cap R} = \psi|_{T \cap S}$$

where $R \cap S = R \times_{y(C)} S$.

More categorically:

Let $\text{Cov}_{\mathcal{J}}(C) \subseteq \text{Sub}(y(C))$ - poset, and consider the canonical functor

$$i_C: \text{Cov}_{\mathcal{J}}(C) \longrightarrow \text{Set}^{\text{cop}}$$

Consider the composite

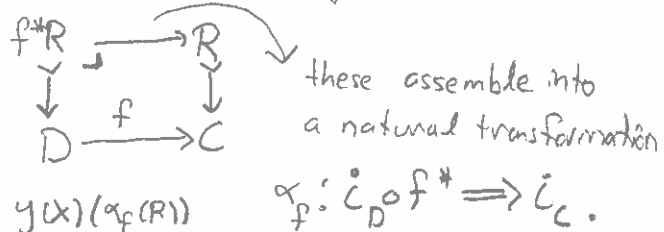
$$(\text{Cov}_{\mathcal{J}}(C))^{\text{op}} \xrightarrow{i_C^{\text{op}}} (\text{Set}^{\text{cop}})^{\text{op}} \xrightarrow{y(X)} \text{Set}$$

$$X^+(C) = \text{colim}_{\mathcal{S} \in \text{Cov}_{\mathcal{J}}(C)} y(X) \circ i_C^{\text{op}}$$

or slightly informally: $X^+(C) = \text{colim}_{\mathcal{S} \in \text{Cov}_{\mathcal{J}}(C)} \text{Hom}(\mathcal{S}, X)$

Need to show that X^+ is actually a presheaf:

$$\text{Let } D \xrightarrow{f} C \rightsquigarrow f^*: \text{Cov}_{\mathcal{J}}(C) \longrightarrow \text{Cov}_{\mathcal{J}}(D)$$



$$\rightsquigarrow \forall R \in \text{Cov}_{\mathcal{J}}(C) \text{ morphisms } \text{Hom}(R, X) \longrightarrow \text{Hom}(f^*R, X).$$

Let $\lambda^D: y(X) \circ i_D^{\text{op}} \implies \Delta_{X^+(D)}$ be the colimiting cocone,

$$\lambda^D(S): \text{Hom}(S, X) \longrightarrow X^+(D)$$

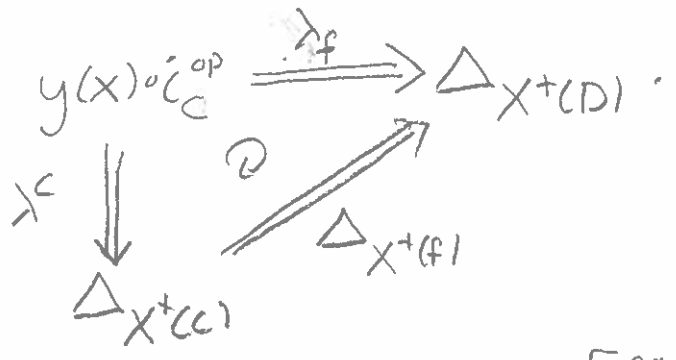
$$\phi: S \rightarrow X \longmapsto [(S, \phi)]$$

then $\exists \lambda_f : y(x) \circ i_c^{op} \implies \Delta_{X^+(D)}$

$$\lambda_f(R) : \text{Hom}(R, X) \xrightarrow{y(x)(\alpha_f(R))} \text{Hom}(f^*R, X) \xrightarrow{\lambda_{f^*R}^D} X^+(D),$$

which is natural since α_f is.

So $\exists ! X^+(f) : X^+(C) \longrightarrow X^+(D)$ s.t.



Concretely: $X^+(f)([R, \psi]) = [f^*R, \psi \circ \alpha_f(R)]$.

Claim: $X \longmapsto X^+$ is a functor $\text{Set}^{eop} \longrightarrow \text{Set}^{eop}$.

Suppose $\theta : X \longrightarrow F$ in Set^{eop} , then $\forall C \in \mathcal{C}_0$ we

have

$$y(x) \circ i_c^{op} \xrightarrow{y(\theta) \circ id} y(F) \circ i_c^{op}$$

$$X^+(C) = \text{colim}_{\longrightarrow} y(x) \circ i_c^{op} \xrightarrow{\theta^+(C)} \text{colim}_{\longrightarrow} y(F) \circ i_c^{op} = F^+(C)$$

$$[R, \phi] \longmapsto [R, \theta \phi]$$

which is easily seen to define a natural transformation

$$\theta^+ : X^+ \implies F^+ \text{ as desired.}$$

Note There is also a canonical natural transformation

$$\sigma : id_{\text{Set}^{eop}} \implies (\cdot)^+$$

$$\begin{array}{ccc}
 X(C) & \xrightarrow{\sim} & \text{Hom}(y(C), X) = \text{Hom}(\text{max}(C), X) \xrightarrow{\lambda_{\text{max}(C)}} X^+(C) \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\sigma_X} & (\tilde{x}: y(C) \rightarrow X) \xrightarrow{\quad} [(\text{max}(C), \tilde{x})],
 \end{array}$$

is natural in C . (follows from naturality of Yoneda), and

$$\sigma_X : X \longrightarrow X^+ \text{ is natural in } X.$$

Prop $(\cdot)^+ : \text{Set}^{\text{eop}} \longrightarrow \text{Set}^{\text{eop}}$ preserves finite limits.

Pf: $\forall C, (\text{Cov}_J(C))^{\text{op}}$ is a filtered poset, and filtered colimits commute with finite limits in Set , hence:

$$\text{If } F = \lim_{\leftarrow} F_i \text{ is a finite limit, } \forall C \in \mathcal{C}_0$$

$$F^+(C) = \text{colim}_{R \in \text{Cov}_J(C)} \text{Hom}(R, F) = \text{colim}_{R \in \text{Cov}_J(C)} \text{Hom}(R, \lim_{\leftarrow} F_i)$$

$$\cong \text{colim}_{R \in \text{Cov}_J(C)} \lim_{\leftarrow} \text{Hom}(R, F_i) \cong \lim_{\leftarrow} \left(\text{colim}_{R \in \text{Cov}_J(C)} \text{Hom}(R, F_i) \right)$$

filtered.

$$\cong \lim_{\leftarrow} (F_i)^+(C).$$

Lemma: Let $X \in \text{Set}^{\text{eop}}$ and $F \in \text{Sh}_J(\mathcal{C})$, and $g: X \rightarrow F$.

Then $\exists!$ \tilde{g} s.t.

$$\begin{array}{ccc}
 X & \xrightarrow{g} & F \\
 \sigma_X \downarrow & \dashrightarrow & \downarrow \tilde{g} \\
 X^+ & &
 \end{array}$$

Pf Let $[(S, \varphi)] \in X^+(C)$,

$$\begin{array}{ccc}
 S & \xrightarrow{\varphi} & X \xrightarrow{g} F \\
 \downarrow & & \downarrow \\
 y(C) & & y(C)
 \end{array}$$

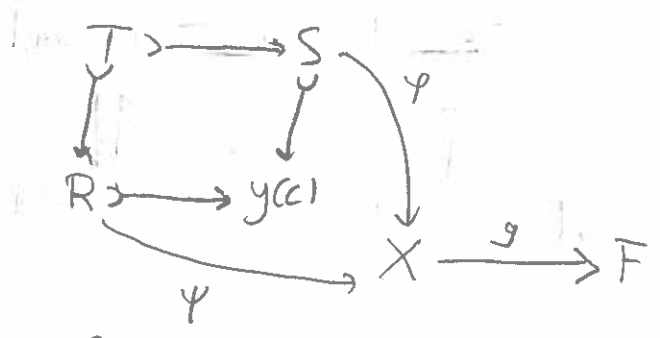
$\exists! \tilde{g}_c(S, \varphi)$

Since S is a covering sieve and F is a sheaf.

Note $\tilde{g}_c(S, \varphi)$ is well defined:

Suppose $(S, \varphi) \sim (R, \psi) \Rightarrow \exists T \in \text{CRNS s.t. } \varphi|_{\text{CRNS}} = \psi|_{\text{CRNS}}$.
cov. sieve.

But $T \in \text{Cov}_S(C)$, so



we have that $\tilde{g}_c(S, \varphi)$ and $\tilde{g}_c(R, \psi)$ restrict to the same map

$T \rightarrow F$, and hence are equal, since T is a cov. sieve and F a sheaf!

(You need F separated)

A similar argument shows that $\tilde{g}_c : X^+(C) \rightarrow \text{Hom}(y(C), F)$

β natural in C .

$$\rightsquigarrow X^+ \xrightarrow{\tilde{g}} \text{Hom}(y(\cdot), F) \cong F.$$

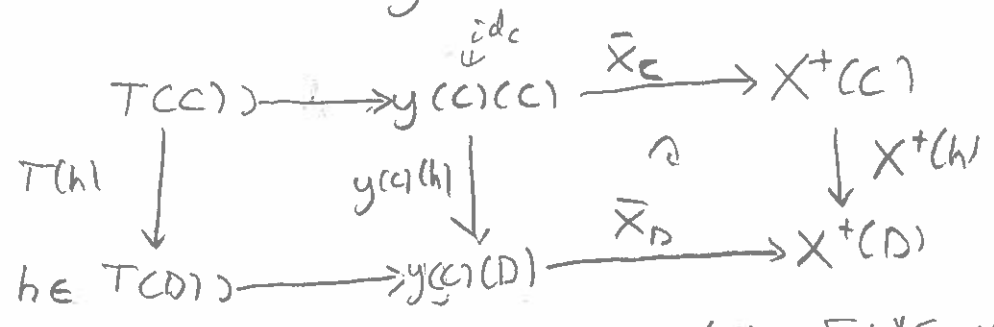
Lemma For any $X \in \text{Set}^{C^{op}}$, X^+ is separated.

Let $T \in \text{Cov}_S(C)$ and consider the map

$$X^+(C) \cong \text{Hom}(y(C), X^+) \xrightarrow{r} \text{Hom}(T, X^+).$$

We want to show that it is mono. Let $[\bar{x}] = [(S, \varphi)]$, $[\bar{y}] = [(R, \psi)] \in X^+(C)$ have the same image under r ,

$$x \longleftarrow \bar{x} : y(C) \rightarrow X^+, \text{ let } (h: D \rightarrow C) \in T(D) :$$



$$r(x) : T \Rightarrow X^+, \quad r(x)(h) = X^+(h)(\bar{x}) = [h^*S, \varphi \circ \alpha_h(S)]$$

So $r(x) = r(y) \Rightarrow \forall h \in T(D) (\forall D)$

5.

$[(h^*S, \varphi \circ \alpha_h(S))] = [(h^*R, \psi \circ \alpha_h(R))] \in X^+(D)$

$\Rightarrow \forall h \exists T_h \in \text{Cov}_J(D)$ with $T_h \subset (h^*S \cap h^*R)$ s.t.

$\forall \varphi \circ \alpha_h(S)|_{T_h} = \psi \circ \alpha_h(R)|_{T_h}$ (†)

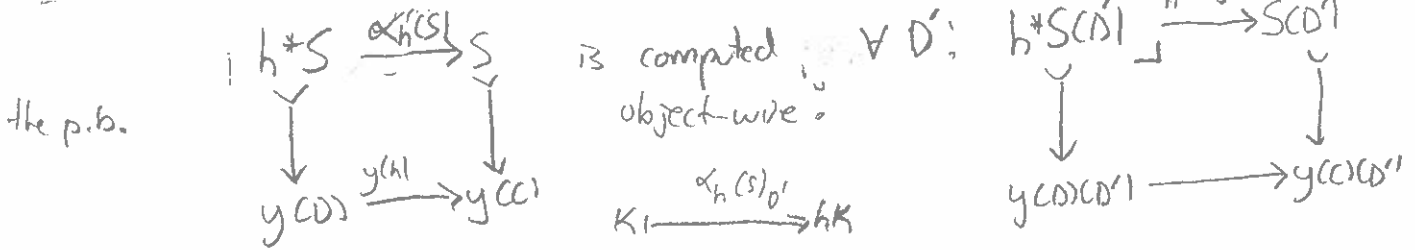
Let $U \rightarrow y(C)$ be the sieve

$U = \{hg : g \in T_h, h \in T(D)\} \in \text{Cov}_J(C)$ and $U \subset R \cap S$

This is a sieve since if $hg : E \rightarrow C, l : E' \rightarrow E$

$(hg)l = h(\underbrace{gl}_{\downarrow T_h}) \in U$. (We already showed it's a covering sieve on page 0)

Note: If $h : D \rightarrow C$ and $S \rightarrow y(C)$



$\Rightarrow h^*(S_{D'}) \cong \{K : D' \rightarrow D \mid hK \in S_{D'}\} \rightarrow S_{D'}$

$$\begin{array}{ccc}
 K & \downarrow & \\
 K & y(h)(D') & \longrightarrow & y(C)(D')
 \end{array}$$

\Rightarrow if $T_h(D') \subseteq h^*(S_{D'}) \cap h^*(R_{D'}) \Rightarrow \forall g \in T_h(D')$
 $hg \in S_{D'} \cap R_{D'}$.

also $\forall g \in T_h(D')$
 $\varphi \circ \alpha_h(S)_{D'}(g) = \varphi(hg) \Rightarrow \varphi|_U = \psi|_U \Rightarrow [R, \psi] = [S, \varphi] \quad \square$
 $\varphi \circ \alpha_h(R)_{D'}(g) = \psi(hg)$

Lemma If X is separated, X^+ is a sheaf.

6.

PF Let $R \in \text{Cov}_J(C)$ and $\phi: R \rightarrow X^+$.

Let $(f: C' \rightarrow C) \in R(C')$ $\xrightarrow{\phi(C')}$ $X^+(C')$,

$$\phi(C') = [(R_f, \phi_f)], \quad R_f \in \text{Cov}_J(C'), \quad \phi_f: R_f \rightarrow X.$$

Let $S \in \text{Cov}_J(C)$ be defined by

$$S = \{fg \mid f \in R, g \in R_f\} \subset R.$$

Define $\psi: S \rightarrow X$ by

$$\begin{array}{ccc} \psi_{C''}: S(C'') & \longrightarrow & X(C'') \\ fg & \longmapsto & (\phi_f|_{C''}(g)). \\ \begin{array}{ccc} C'' \xrightarrow{g} C' \xrightarrow{f} C \\ \downarrow \quad \downarrow \\ R_f \quad R \end{array} & & \end{array}$$

It is not clear yet that ψ is natural, or even well defined. We will show this in a minute. For now, assume this has been shown.

Note that $r: \text{Hom}(y(C), X^+) \rightarrow \text{Hom}(R, X^+)$ is the restriction map,

then $r([S, \psi])|_R: R \rightarrow X^+$, by the previous proof is of the form

$$\begin{array}{ccc} r([S, \psi])_{C'}: R(C') & \longrightarrow & X^+(C') \\ \downarrow \psi & \longmapsto & [f^*S, \psi \circ \alpha_f(S)], \text{ and} \end{array}$$

$$\begin{array}{ccc} (f^*S)(C'') = \{K: C'' \rightarrow C' \mid fK \in S\} & \xrightarrow{\psi(C')} & S(C') \\ K \longmapsto fK & & \downarrow \psi_{C'} \\ \star & \searrow & X(C'') \\ & & (\phi_f)_{C''}(K). \end{array}$$

Note: if $(K: C'' \rightarrow C') \in R_f(C')$ $\Rightarrow fK \in S$ (by def'n)
 $\Rightarrow R_f \subset f^*S \quad \forall f$ and $\star \Rightarrow (\psi_{C''} \circ \alpha_f(S))|_{R_f} = \phi_f$.

$$\circ. \quad r([S, \psi])_{C'}(f) = [(R_f, \phi_f)] = \phi_{C'}(f) \quad \forall f \Rightarrow r([S, \psi]) = \phi. \quad 7.$$

Since ϕ was arbitrary $\Rightarrow r$ is surjective, and X^+ is separated $\Rightarrow r$ is mono, $\therefore r$ is an isomorphism, and X^+ is a sheaf.

It now suffices to show that ψ is well-defined & natural.

Well-defined:

Suppose $f, f' \in R, g \in R_f, g' \in R_{f'}$ s.t.

$$\begin{array}{ccc} C'' & \xrightarrow{g} & C' \\ & \searrow & \downarrow f \\ & & C \\ & \swarrow g' & \uparrow f' \\ & D & \end{array} \quad fg = f'g' \in R_{C''}$$

WTS $(\phi_f)_{C''}(g) = (\phi_{f'})_{C''}(g')$.

Naturality for $\phi \Rightarrow$

$$\begin{array}{ccc} f \in R_{C'} & \xrightarrow{\phi_{C'}} & X^+(C) \\ \downarrow R(g) & & \downarrow X^+(g) \\ fg \in R_{C''} & \xrightarrow{\phi_{C''}} & X^+(C'') \end{array}$$

$$\begin{aligned} \text{so } \phi_{C''}(fg) &= X^+(g)(\phi_{C'}(f)) = X^+(g)([R_f, \phi_f]) \\ &= [g^*R_f, \phi_f \circ \alpha_g(R_f)] \end{aligned}$$

$$\Rightarrow [g^*R_f, \phi_f \circ \alpha_g(R_f)] = [g'^*R_{f'}, \phi_{f'} \circ \alpha_{g'}(R_{f'})] \Rightarrow$$

$$\exists \tau \in \text{Cov}_\gamma(C''), \tau \subseteq g^*R_f \cap g'^*R_{f'} \text{ s.t.}$$

$$\phi_f \circ \alpha_g(R_f)|_\tau = \phi_{f'} \circ \alpha_{g'}(R_{f'})|_\tau \quad (**)$$

But note: $(g^*R_f)(E) = \{ \gamma: E \rightarrow C'' \mid g\gamma \in R_f \}$

$$\begin{array}{ccc} \xrightarrow{\alpha_g(R_f/E)} & R_f(E) & \xrightarrow{(\phi_f)(E)} \\ \gamma \longmapsto & g\gamma & \rightarrow X(E) \end{array}$$

$$\text{so } (***) \Rightarrow \forall \gamma \in \tau(E) \quad (\phi_f)(g\gamma) = (\phi_{f'})(g'\gamma).$$

Consider the naturality square

$$\begin{array}{ccc}
 g \circ R_f(C^c) & \xrightarrow{(\phi_f)_{C^c}} & X(C^c) \\
 \downarrow R_f(\gamma) & \circlearrowleft & \downarrow X(\gamma) \\
 R_f(E) & \xrightarrow{(\phi_f)_E} & X(E)
 \end{array}
 \Rightarrow (\phi_f)_{C^c}(g) = X(\gamma)(\phi_f)_E(g)$$

hence: $\forall \gamma \in T, X(\gamma)(\phi_f)_{C^c}(g) = X(\gamma)(\phi_{f'})_{C^c}(g')$. (***)

Note: Since X is separated, the canonical map

$$\begin{aligned}
 \Gamma_T : X(C^c) &\longrightarrow \text{Hom}(T, X) \\
 \gamma &\longmapsto \beta(\gamma) : T \Rightarrow X \quad (\gamma : E \rightarrow C^c) \\
 & \\
 \beta(\gamma)(E) : T(E) &\longrightarrow X(E) \\
 \gamma &\longmapsto X(\gamma)(\gamma).
 \end{aligned}$$

is mono.

So, by (***) $\Rightarrow (\phi_f)_{C^c}(g) = (\phi_{f'})_{C^c}(g')$.

Finally, need to show that $\Psi_{C^c} : S(C^c) \rightarrow X(C^c)$ is natural.

$$\begin{array}{ccc}
 f \circ g & \longmapsto & (\phi_f)_{C^c}(g) \\
 \hline
 \text{Suppose } h : C^c \rightarrow C^c, \text{ WTS } & S(C^c) & \xrightarrow{\Psi_{C^c}} X(C^c) \\
 & \downarrow S(h) & \circlearrowleft \downarrow X(h) \\
 & S(C^c) & \xrightarrow{\Psi_{C^c}} X(C^c)
 \end{array}$$

WTS: $\Psi_{C^c}(f \circ g) = X(h)(\Psi_{C^c}(f \circ g))$

$\hookrightarrow (\phi_f)_{C^c}(gh)$

Note $\phi_f : R_f \rightarrow X$ is natural

$$\begin{array}{ccc}
 g & \circlearrowleft & \downarrow X(h) \\
 R_f(C^c) & \xrightarrow{(\phi_f)_{C^c}} & X(C^c) \\
 \downarrow R_f(h) & \circlearrowleft & \downarrow X(h) \\
 R_f(C^c) & \xrightarrow{(\phi_f)_{C^c}} & X(C^c) \\
 \uparrow gh & & \\
 R_f(C^c) & \xrightarrow{(\phi_f)_{C^c}} & X(C^c)
 \end{array}$$

□

Cor The inclusion $i: Sh_J(\mathcal{C}) \hookrightarrow Set^{eop}$

admits a left exact left adjoint $a \dashv i$.

Pf If X is arbitrary, X^+ is separated \Rightarrow

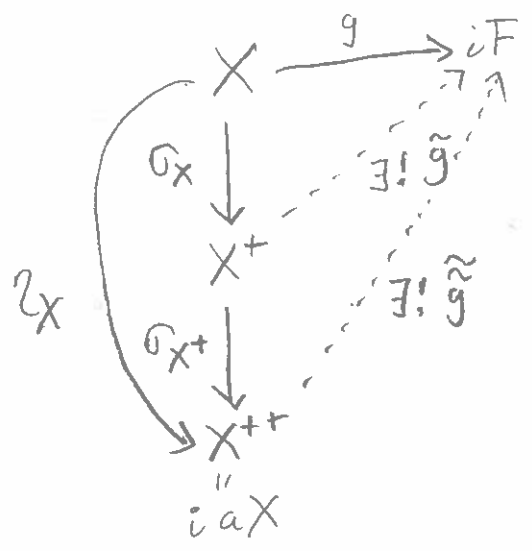
X^{++} is always a sheaf. By restriction of codomain,

$$a := (\cdot)^+ \circ (\cdot)^+ : Set^{eop} \rightarrow Sh_J(\mathcal{C}).$$

There is a canonical natural transformation

$$\eta : id \Rightarrow a \text{ given by } \eta_X = (X \xrightarrow{\sigma_X} X^+ \xrightarrow{\sigma_{X^+}} X^{++} = aX).$$

Now, if $F \in Sh_J(\mathcal{C})$ and $g: X \rightarrow iF$



\Rightarrow composition with η_X induces a bijection

$$Hom(iX, F) \cong Hom(X, iF)$$

$\Rightarrow \eta$ is the unit of an adjunction

$a \dashv i$.

Finally, since each $(\cdot)^+$ preserves finite limits, so does a .