Reminder: (Lecture 4)

Last time: the functor $j : \mathcal{O}(X) \longrightarrow \mathrm{Top}/X$

$$U \longmapsto U \hookrightarrow X$$

$$\rightsquigarrow \quad \mathrm{Set}^{\mathcal{O}(X)^{op}} \underset{\Lambda = \mathrm{Lan}_y j}{\overset{R_j = \Gamma}{\rightleftarrows}} \mathrm{Top}/X \, , \quad \Lambda \dashv \Gamma.$$

$$R_j(P \xrightarrow{\pi} X)(U) = \mathrm{Hom}(j(U), \pi) = \left\{ \begin{tikzcd} & P \arrow[d, "\pi"] \\ U \arrow[ur, dashed, "\sigma"] \arrow[r, hook] & X \end{tikzcd} \right\} = \Gamma(\pi)(U)$$

So Γ is the <u>sheaf of sections functor</u>.

Given $F \in \hat{\mathcal{O}}(X)$, we have $\mathcal{O}(X)/F \xrightarrow{\pi_F} \mathcal{O}(X)$ and

$$F = \varinjlim y \circ \pi_F .$$

Λ is the unique colimit pres. functor s.t. $\Lambda \circ y \cong j$, so

$$\Lambda(F) \cong \varinjlim \Lambda \circ y \circ \pi_F \cong \varinjlim j \circ \pi_F .$$

$\Lambda(F)$ is called the <u>étalé space</u> of F.

<u>Notation</u> $\Lambda(F) = (E(F) \xrightarrow{e_F} X) \in \mathrm{Top}/X.$

<u>Recall</u>

$$\mathrm{Set}/UX \underset{\tilde{\Psi}}{\overset{\tilde{\varphi}}{\rightleftarrows}} \mathrm{Set}^{|UX|}$$

$$(S \xrightarrow{f} UX) \overset{\Psi}{\longmapsto} (f^{-1}(x), x \in X)$$

$$\left(\coprod_x A_x \longrightarrow UX\right) \overset{\varphi}{\longleftarrow\!\!\!\shortmid} (A_x, x \in X)$$

$(\forall x \in X)$

We have the composite

$$\begin{tikzcd} \mathrm{Set}^{\mathcal{O}(X)^{op}} \arrow[r, "\Delta"] & \mathrm{Top}/X \arrow[r, "U/X"] \arrow[rrr, bend right, "\mathrm{fib}_x"'] & \mathrm{Set}/UX \arrow[r, "\Psi", "\sim"'] & \mathrm{Set}^{|UX|} \arrow[r, "ev_x"] & \mathrm{Set} \end{tikzcd}$$

$(A_y, y \in X) \longmapsto A_x$

fib_x (colimit preserving)

Last time: $\mathrm{fib}_x \circ \Delta(F) \cong F_x$ (stalk).

<u>Now</u> Since $\Psi \circ U/X$ is colimit preserving

$$\Psi \circ U/X \circ \Delta = \Psi \circ (U/X) \circ \mathrm{Lan}_y j \cong \mathrm{Lan}_y (\Psi \circ (U/X) \circ j)$$

$$\Psi \circ (U/X) \circ \Delta(F) = \varinjlim (\Psi \circ (U/X) \circ j \circ \pi_F) \leftarrow \text{colimit in } \mathrm{Set}^{|UX|}$$

so computed ptwise

$$\therefore \quad \cong (\varinjlim \underbrace{ev_x \circ \Psi \circ (U/X)}_{\mathrm{fib}_x} \circ j \circ \pi_F, x \in X)$$

$\updownarrow$ fib_x pres. colims

$$\cong (\mathrm{fib}_x \circ \varinjlim j \circ \pi_F, x \in X)$$

$$\cong (\mathrm{fib}_x \circ \Delta(F), x \in X)$$

$$\cong (F_x, x \in X).$$

$$\Rightarrow (U/X) \circ \Delta(F) \cong \Psi \circ \Psi (U/X \circ \Delta(F)) \cong \Psi(F_x, x \in X)$$

$$\cong (\coprod_x F_x \longrightarrow X).$$

<u>Observe</u>: $U/X : \mathrm{Top}/X \longrightarrow \mathrm{Set}/UX$ creates colimits.

What is the colimiting cocone $\rho : U/X \circ j \circ \pi_F \Longrightarrow \Delta_{\underline{\Lambda(F)}}$,

where $\pi_F : \mathcal{O}(X)/F \longrightarrow \mathcal{O}(X)$?

For $(\lambda : y(V) \longrightarrow F) \in \mathcal{O}(X)/F$

$$\begin{tikzcd} y \in V \arrow[r, "\rho(\lambda)"] \arrow[dr] & \coprod_{x \in X} F_x \arrow[d, "e_F"] \\ & X \end{tikzcd}$$

, where

$\rho(\lambda)(y) = \mathrm{germ}_y \tilde{\lambda} \in F_y$

Where $\mathrm{germ}_y : F \circ i_y \Longrightarrow \Delta_{F_y}$ is the colimiting cocone expressing

$F_y = \varinjlim_{y \in U} F(U)$, and $\tilde{\lambda} \in F(V) \underset{\text{Yoneda}}{\overset{\sim}{\longrightarrow}} \lambda : y(V) \longrightarrow F$ (i.e. $\lambda_V(\mathrm{id}_V)$).

It follows that $P_X(\underset{\overset{\|}{E(F)}}{\Lambda(F)})$ is $\coprod_{x \in X} F_x$ equipped with

the final topology wrt the maps $\left\{ \rho(\lambda) : V \longrightarrow \coprod_{x \in X} F_x \;\middle|\; \begin{matrix} \lambda : y(V) \longrightarrow F \\ V \in \mathcal{O}(X) \end{matrix} \right\}$.

Denote this topological space by $E(F)$.

<u>Prop</u> The projection map $E(F) = \coprod_{x \in X} F_x \xrightarrow{e_F} X$ is

a local homeomorphism.

<u>Pf</u> 1[st] continuity: Note $e_F \circ \rho(\lambda) = V \hookrightarrow X \quad \forall$

$\lambda : y(V) \longrightarrow F$, so $e_F \circ \rho(\lambda)$ is cont. $\forall \lambda \Rightarrow e_F$ is continuous.

2nd/ $\forall\ \lambda : y(V) \longrightarrow F$,

$\rho(\lambda)(V) \subset E(F)$ is open:

$\Leftrightarrow \forall\ \mu : y(U) \longrightarrow F$, $\rho(\mu)^{-1}\rho(\lambda)(V)$ is open in U

$$\rho(\mu)^{-1}\rho(\lambda)(V) = \{x \in U \mid \rho(\mu)(x) \in \rho(\lambda)(V)\}$$

$$= \{x \in U \mid \exists\, y \in V \text{ s.t. } \rho(\mu)(x) = \rho(\lambda)(y)\}$$

must be in the same stalk F_x, so $y = x$ $\Rightarrow x \in U \cap V$

$$= \{x \in U \cap V \mid \text{germ}_x \hat{\mu} = \text{germ}_x \hat{\lambda}\}$$

$\hookrightarrow$ $\hat{\mu}$ and $\hat{\lambda}$ agree on some open of x

$\therefore$ open ✓.

3rd/ Now let $z \in E(F) \xrightarrow{e_F} X \Rightarrow z \in F_x \Rightarrow z = \text{germ}_x \alpha$

$z \longmapsto x$

for some $\alpha \in F(U)$, $x \in U$. $\alpha \overset{\text{Yoneda}}{\rightsquigarrow} \hat{\alpha} : y(U) \longrightarrow F$

and $\rho(\hat{\alpha})(U) \ni z$ is an open nbd.

Claim: $e_F|_{\rho(\hat{\alpha})(U)} : \rho(\hat{\alpha})(U) \longrightarrow U$ and $\rho(\hat{\alpha}) : U \longrightarrow \rho(\hat{\alpha})(U)$

are continuous and inverse to another. ☑.

$\rho(\hat{\alpha})(y) = \text{germ}_y \alpha \Rightarrow$
$e_F\, \rho(\hat{\alpha})(y) = y$ so $e_F \circ \hat{\rho}(\alpha) = i$

$w \in \rho(\hat{\alpha})(U) \Rightarrow w = \text{germ}_{x'} \alpha$
$x' \in U \Rightarrow \hat{\rho}(\alpha) \circ e_F(w)$
$= \hat{\rho}(\alpha) \circ e_F(\text{germ}_{x'} \alpha) = \hat{\rho}(\alpha)(x')$
$= \text{germ}_{x'} \alpha = w \Rightarrow \hat{\rho}(\alpha) \circ e_F|_{w} = id$

$$\mathbf{Set}^{\mathcal{O}(X)^{op}} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Top}/X$$

$\Delta \dashv \Gamma$.

What are unit and co-unit? Following HW2:

<u>Unit</u> $F \overset{\eta_F}{\Longrightarrow} \Gamma\Delta F$, $\alpha \rightsquigarrow \hat{\alpha} : y(U) \longrightarrow F$

$U \xrightarrow{\rho(\hat{\alpha})} E(F) \Rightarrow \rho(\hat{\alpha}) \in \Gamma\Delta F(U)$

$$\begin{tikzcd} U \arrow[r, "\rho(\hat{\alpha})"] \arrow[dr, bend right] & E(F) \arrow[d, "e_F"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd}$$

$\alpha \in F(U) \xrightarrow{\eta_F(U)} \Gamma\Delta F(U)$

$\alpha \longmapsto \rho(\hat{\alpha})$.

<u>Lemma</u>: F is a sheaf $\iff \eta_F$ is an isomorphism.

Pf $\Gamma(\pi)$ is always a sheaf $\forall \pi$ so η_F iso $\implies F$ is a sheaf.

Suppose F is a sheaf.

$$F(U) \xrightarrow{\eta_F(U)} \Gamma\Delta F(U)$$

$$\alpha \longmapsto \left(U \xrightarrow{\rho(\alpha)} E(F)\right)$$

$$x \longmapsto \mathrm{germ}_x \alpha$$

$\eta_F(\alpha) = \eta_F(\beta) \iff \mathrm{germ}_x \alpha = \mathrm{germ}_x \beta \quad \forall x \in U$

$\iff \alpha$ and β have the same image in $\prod_{x \in X} F_x$

$\iff \alpha = \beta$ since F is seperated. (from HW 3)

$\implies \eta_F(U)$ is injective $\forall U$.

Lets show it is surjective. Let $\sigma \in \Gamma\Delta F(U)$.

$$\begin{tikzcd} & E(F) \arrow[d, "e_F"] \\ U \arrow[ur, "\sigma"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

$\forall x \in U$ $\exists$ a nbd $U_x \ni x$ and $s^x \in F(U_x)$ s.t.
$\sigma(x) = \mathrm{germ}_x s^x$. (since $\sigma(x) \in F_x$)

$\implies \hat{\rho}(\hat{s}^x)|_{U \cap U_x}$ & $\sigma|_{U \cap U_x}$ are sections of e_F over $U \cap U_x$

that agree at x $\therefore$ $\exists$ a nbd $W_x \ni x$ s.t.

$\hat{\rho}(\hat{s}^x)|_{W_x} = \sigma|_{W_x}$. (since e_F is a local homeo)

Let $\mathcal{W} = (W_x, x \in X)$ cover of U.

<u>Note</u>: $\rho(\hat{s}^x)|_{W_x \cap W_y} = \sigma|_{W_x \cap W_y} = \rho(\hat{s}^y)|_{W_x \cap W_y}$

$\eta_F(s^x|_{W_x \cap W_y}) \qquad \eta_F(s^y|_{W_x \cap W_y})$

η_F injective $\implies s^x|_{W_x \cap W_y} = s^y|_{W_x \cap W_y} \quad \forall x, y$

Since F is a sheaf, $\exists!\ s \in F(U)$ s.t.

$$s|_{W_x} = s^x|_{W_x} \quad \forall\, x \in U.$$

<u>Claim</u>: $\sigma = \eta_F(s)$.

Since $(W_x, x \in U)$ is a cover of U, and σ and $\rho(\hat{s})$ are both sections of e_F over U, it suffices to show that

$\sigma|_{W_x} = \rho(\hat{s})|_{W_x}\ \forall x$ since this $\Rightarrow \sigma = \rho(\hat{s}) = \eta_F(s)$.

By naturality:

$$\sigma|_{W_x} = \rho(\hat{s}^x)|_{W_x} = \rho\Big(\overset{\widehat{F_1(W_x \hookrightarrow U_x)(s)} \\ \shortparallel}{\widehat{s^x|_{W_x}}}\Big) = \rho(\widehat{s|_{W_x}})$$

$$= \rho(\hat{s})|_{W_x}.$$

$\boxtimes$

<u>Counit</u> $\varepsilon_\pi : \Delta\Gamma(\pi) \longrightarrow \pi$ in Top/X,

from HW2 we know the counit comes from:

$$\Delta\Gamma(\pi) = (\mathrm{Lan}_y j)(\Gamma(\pi)) = \underset{\lambda : y(U) \to \Gamma(\pi)}{\underrightarrow{\mathrm{colim}}}\,(U \hookrightarrow X) \underset{\text{Yoneda}}{\cong} \underrightarrow{\mathrm{colim}}\,(U \hookrightarrow X)$$

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ U \arrow[ur, dashed, "\tilde{\lambda}"] \arrow[r, hook] & X \end{tikzcd}$$

or formally, $\Delta\Gamma(\pi) = \underrightarrow{\mathrm{colim}}\ j \circ \Pi_{\Gamma(\pi)}$, where

$$\Pi_{\Gamma(\pi)} : \mathcal{O}(X)/\Gamma(\pi) \longrightarrow \mathcal{O}(X);$$

The co-unit is induced by the canonical cocone

$$\xi : j \circ \Pi_{\Gamma(\pi)} \Longrightarrow \Delta_\pi$$

$$\xi(\lambda : y(U) \to \Gamma(\pi)) = \begin{tikzcd} & P \arrow[d, "\pi"] \\ \underset{j \circ \lambda}{U} \arrow[ur, "\tilde{\lambda}"] \arrow[r, hook] & X \end{tikzcd}. \quad (*)$$

Since $\Delta\Gamma(\pi) = \underrightarrow{\mathrm{colim}}\ j \circ \Pi_{\Gamma(\pi)}$, $\xi \rightsquigarrow\, !\ \varepsilon_\pi : \Delta\Gamma(\pi) \longrightarrow \pi$,

Which is a map of the form

$$\begin{tikzcd} E\Gamma(\pi) \arrow[rr, "\varepsilon_\pi"] \arrow[dr, "e_{\Gamma(\pi)}"'] & \arrow[d, phantom, "\circlearrowleft"] & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}.$$

Since $p_X : \mathrm{Top}/X \longrightarrow \mathrm{Top}$ preserves colimits, <u>ε_π</u> is induced by the cocone $p_X \xi$.

Explicitly:

If $z \in E\Gamma(\pi)$, with $e_{\Gamma(\pi)}(z) = x$ (i.e. $z \in \Gamma(\pi)_x$) $\Rightarrow \exists_{x \in U}$
$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ U \arrow[ur, dashed, "\tilde{\lambda}"] \arrow[r, hook] & X \end{tikzcd}$$
s.t.

$z = \mathrm{germ}_x \tilde{\lambda}$, and <u>$\varepsilon_\pi$</u>$(z) = \tilde{\lambda}(x)$. $(*)$

$$\left(\begin{array}{c} \Gamma(\pi)_x \xrightarrow{\varepsilon_{\pi_x}} \pi^{-1}(x) \\ \| \\ \underset{x \in U}{\underrightarrow{\mathrm{colim}}}\ \Gamma(\pi)(U) \end{array} \right)$$

Lemma: For $\pi : P \to X \in Top/X$, the counit ε_π is an iso. $\Leftrightarrow$ π is a local homeomorphism. (sometimes called an étale map).

Pf If ε_π is an iso, then $\Lambda\Gamma(\pi)$ loc. homeo $\Rightarrow$ π is too. (HW 3)

Conversely, suppose π is a local homeo:

$$\begin{tikzcd} E\Gamma(\pi) \arrow[rr, "\varepsilon_\pi"] \arrow[dr, "e_{\Gamma(\pi)}"', "l.h."] & & P \arrow[dl, "\pi", "l.h."'] \\ & X & \end{tikzcd} \Rightarrow \varepsilon_\pi \text{ is too. (HW 3)}$$

To show ε_π is a homeo., suffices to show it's bijective. (HW 3) Define a set-theoretic map $\Theta : P \longrightarrow E\Gamma(\pi) = \coprod_x \Gamma_x(\pi)$ as follows:

Given $p \in P$, choose an open nbd $p \in W_p \xrightarrow[\sim]{\pi|_{W_p}} \pi(W_p) \ni \pi(p) =: x$.
homeo.

Let $\underset{\Gamma(\pi)_x}{\underset{\cap}{\Theta(p) := \mathrm{germ}_x(\pi|_{W_p}^{-1})}}$ $\left(\begin{tikzcd} & P \arrow[d, "\pi"] \\ \pi(W_p) \arrow[ur, "\pi|_{W_p}^{-1}"] \arrow[r, hook] & X \end{tikzcd} \in \Gamma(\pi)(W_p) \right)$.

Easy to check that if $p \in W_p' \xrightarrow[\text{homeo.}]{\pi|_{W_p'}} \pi(W_p')$, $\mathrm{germ}_x(\pi|_{W_p}^{-1}) = \mathrm{germ}_x(\pi|_{W_p'}^{-1})$.

Now $\Theta(p) = \mathrm{germ}_x(\pi|_{W_p}^{-1}) \in E\Gamma(\pi)$, so by definition

$$\varepsilon_\pi(\Theta(p)) = \pi|_{W_p}^{-1}(x) = p, \text{ so } \quad \varepsilon_\pi \circ \Theta = id_P.$$

Conversely, if $z = \mathrm{germ}_x \tilde{\lambda} \in E\Gamma(\pi)$, with $\begin{tikzcd} & P \arrow[d, "\pi"] \\ U \arrow[ur, "\tilde{\lambda}"] \arrow[r, hook] & X \end{tikzcd}$, then

$$\Theta\varepsilon_\pi(z) = \Theta(\tilde{\lambda}(x)).$$

Note: $\tilde{\lambda}(U)$ is open and $\tilde{\lambda}(x) \in \tilde{\lambda}(U) \xrightarrow[\sim]{\pi|_{\tilde{\lambda}(U)}} \pi(\tilde{\lambda}(U)) = U$
homeo

$\Rightarrow$ $\Theta(\tilde{\lambda}(x)) = \mathrm{germ}_x(\pi|_{\tilde{\lambda}(U)}^{-1}) = \mathrm{germ}_x(\tilde{\lambda}) = z$ $\Rightarrow$ $\Theta \circ \varepsilon_\pi = id_{E\Gamma(\pi)}$.

$\boxtimes$

<u>Theorem</u>: The adjunction

$$\mathrm{Set}^{\mathcal{O}(X)^{op}} \underset{\Lambda}{\overset{\Gamma}{\leftrightarrows}} \mathrm{Top}/X \; ; \quad \Lambda \dashv \Gamma$$

restricts to an adjoint eq'l of categories

$$\mathrm{Sh}(X) \underset{\bar{\Lambda}}{\overset{\bar{\Gamma}}{\leftrightarrows}} \mathcal{E}t/X,$$

where $\mathcal{E}t/X$ is the full subcategory of Top/X spanned by those $\pi : P \to X$ s.t. π is a local homeomorphism.

<u>Pf</u>: From HW2, we know that the adj. $\Lambda \dashv \Gamma$ restricts to an adjoint eq'l between the subcat of $\mathrm{Set}^{\mathcal{O}(X)^{op}}$ on which the unit η is an iso., and the subcat of Top/X on which the counit is an iso. - but these are $\mathrm{Sh}(X)$ and $\mathcal{E}t/X$ respectively.

<u>Corollary</u>: The full subcat $\mathrm{Sh}(X)$ of $\mathrm{Set}^{\mathcal{O}(X)^{op}}$ is reflective, i.e. the full and faithful inclusion

$$\mathrm{Sh}(X) \xhookrightarrow[i]{} \mathrm{Set}^{\mathcal{O}(X)^{op}} \text{ admits a left adjoint.}$$

<u>Pf</u> i is canonically naturally isomorphic to the composite

$$\mathrm{Sh}(X) \xrightarrow[\bar{\Lambda}]{\sim} \mathcal{E}t/X \underbrace{\xrightarrow{\Gamma_{\mathcal{E}t/X}}}_{\substack{\text{f\&f since} \\ \text{co-unit is an iso.}}} \mathrm{Set}^{\mathcal{O}(X)^{op}}, \text{ which}$$

has a left adjoint

co-restricted

$$\mathrm{Sh}(X) \underset{\sim}{\overset{\bar{\Gamma}}{\longleftarrow}} \mathcal{E}t/X \overset{\Lambda^{\mathcal{E}t/X}}{\longleftarrow} \mathrm{Set}^{\mathcal{O}(X)^{op}}.$$

$$\mathrm{Hom}_{\mathrm{Set}^{\mathcal{O}(X)^{op}}}(F, iG) \cong \mathrm{Hom}_{\mathrm{Set}^{\mathcal{O}(X)^{op}}}(F, \Gamma\Lambda i G) \cong \mathrm{Hom}_{\mathrm{Top}/X}(\Lambda F, \Lambda i G) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(\Gamma\Lambda F, iG).$$