Sheaves on Spaces — Lecture 2

Def Let X be a top'l space. A presheaf on X is a functor $F : \mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$, where $\mathcal{O}(X)$ is the poset of open subsets, i.e.

$$U \subset V \rightsquigarrow \iota_{V,U} : U \longrightarrow V \quad (\&\text{ such an arrow is unique}).$$

I.e., a presheaf on X is a presheaf on $\mathcal{O}(X)$.

Example

$$C(\cdot, \mathbb{R}) : \mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$$
$$U \longmapsto C(U,\mathbb{R}) = \{ f : U \to \mathbb{R},\ f \text{ cont.} \}$$

if $U \hookrightarrow V$,

$$\rightsquigarrow C(\iota_{V,U}, \mathbb{R}) : C(V,\mathbb{R}) \longrightarrow C(U,\mathbb{R})$$
$$f \longmapsto f|_U .$$

Notice: Let $(U_i \hookrightarrow U)_{i\in I}$ form an open cover of U.

Consider the set

$$\varprojlim \Big(\prod_{i\in I} C(U_i,\mathbb{R}) \underset{\text{induced by } U_i\cap U_j \hookrightarrow U_j}{\overset{\text{induced by } U_i\cap U_j \hookrightarrow U_i}{\rightrightarrows}} \prod_{i,j} C(U_i\cap U_j, \mathbb{R}) \Big) \qquad \left((f_i)_{i\in I} \begin{array}{l} \mapsto (f_i|_{U_i\cap U_j}) \\ \mapsto (f_j|_{U_i\cap U_j}) \end{array} \right)$$

$$= \{ (f_i)_{i\in I} \in C(U_i,\mathbb{R}) \mid \forall\, i,j \ \ f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j} \}$$

There is a canonical map

$$C(U,\mathbb{R}) \longrightarrow \varprojlim \left(\prod_i C(U_i,\mathbb{R}) \rightrightarrows \prod_{i,j} C(U_{ij},\mathbb{R}) \right) \quad (*)$$

$$f \longmapsto (f|_{U_i})_i$$

Note: If f & g are s.t. $f|_{U_i} = g|_{U_i}$ $\forall i \Rightarrow f = g$

$\Rightarrow$ $(*)$ is a monomorphism.

Also, given a collection of continuous functions $f_i : U_i \longrightarrow \mathbb{R}$ s.t. $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ $\forall\, i,j$, by continuity, $\exists!\ f : U \longrightarrow \mathbb{R}$ s.t. $f|_{U_i} = f_i$ $\forall\, i$,

so $(*)$ is an iso.

$C(\cdot,\mathbb{R})$ is a prototypical example of a sheaf.

<u>Def</u> A presheaf F on X is a <u>sheaf</u> iff $\forall\ U \in \mathcal{O}(X)$ and $\forall\ (U_i \hookrightarrow U)_i$ an open cover, the canonical map

$$F(U) \longrightarrow \varprojlim \left[\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I^2} F(U_{ij}) \right] \quad (**)$$

is an isomorphism. ~~A~~ A presheaf F is called <u>separated</u> if $(**)$ is a mono.

Denote the full subcategory of $Set^{\mathcal{O}(X)^{op}}$ on the sheaves, by $Sh(X)$.

Note: If F is a sheaf, $F(\phi)$ is terminal.

Pf Consider the empty cover of ϕ (i.e. a cover indexed over the empty set).

The products in (**) are then all indexed over the empty set, so are empty products, hence $\cong$ terminal object.

So we get $F(\phi) \xrightarrow{\sim} \varprojlim\{* \rightrightarrows *\} \cong *$.

There's nothing so special about $\mathbb{R}$:

If Y is any top'l space, $C(\cdot, Y)$ is also a sheaf, by the same argument.

Ex: More generally, let $P \xrightarrow{\pi} X$ be any continuous map.

Let
$$\Gamma(\pi) : \mathcal{O}(X)^{op} \longrightarrow X$$
$$U \longmapsto \Gamma(\pi)(U) = \left\{ \begin{tikzcd} & P \arrow[d, "\pi"] \\ U \arrow[ur, "f"] \arrow[r, hook] & X \end{tikzcd} \right\}$$

This is also a sheaf.

Rmk Can recover $C(\cdot, Y)$ as $\Gamma(Y \times X \xrightarrow{pr} X)$.

Ex Let M & N be smooth mfds. Then
$$C^k(\cdot, N) : \mathcal{O}(M)^{op} \longrightarrow Set$$
$$U \longmapsto \{f : U \to N \text{ of class } C^k\}$$
is a sheaf, $\forall k$.

Ex Let $\pi : P \to M$ be smooth. Then
$$\Gamma^k(\pi) : \mathcal{O}(M)^{op} \longrightarrow Set$$
$$U \longmapsto \{C^k\text{-sections of } \pi \text{ over } U\}$$
is a sheaf $\forall k$.

E.g. $\pi : TM \longrightarrow M$

$\Gamma^k(\pi)(U) = \mathfrak{X}^k(U) = C^k$-diff'l v.f.s on U

or $\pi : T^*M \longrightarrow M \rightsquigarrow$ 1-forms

etc. etc.

$\Omega^n(\cdot)$ = differential n-forms are also a sheaf.

(Basically any type of geometric structure gives a sheaf, e.g. Riemannian metrics, symplectic forms ... since everything is defined by coherent local data)

Let Y be another tp'l space

Ex: $\mathrm{Emb}(\cdot, Y): \mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$

$U \longmapsto \{\text{abstract embeddings } U \hookrightarrow Y\}$

↑ treating U as its own space.

If φ, ψ are two embeddings of $U \hookrightarrow Y$ and $(\iota_{U,U_i}: U_i \hookrightarrow U)_i$ is an open covering, s.t. $\varphi|_{U_i} = \psi|_{U_i}\ \forall i \Rightarrow \varphi = \psi$. (i.e. $\bigcup_i U_i = U$) So $\mathrm{Emb}(\cdot, Y)$ is separated

But, suppose $f: U \hookrightarrow Y$ is not an embedding, but a local homeomorphism, e.g. $U = X = \mathbb{R}$, $Y = S^1$; f = cov. proj.

Then $\exists$ a cover U_i of U s.t. $f_i := f|_{U_i} : U_i \longrightarrow Y$ is an embedding $\forall i$. f is the unique function $U \longrightarrow Y$ s.t. $f|_{U_i} = f_i\ \forall i$ (since $C(\cdot, Y)$ is separated), and f is not an embedding, $\therefore$ $\mathrm{Emb}(\cdot, Y)$ is not a sheaf.

Ex Similarly, $B(\cdot): \mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$

$U \longmapsto \{f: U \to \mathbb{R} \mid f \text{ is bounded}\}$

is not (in general) a sheaf, but is separated.

But $B^{loc}(\cdot): \mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$

$U \longmapsto \{f: U \to \mathbb{R} \mid f \text{ is \underline{locally} bdd}\}$

is a sheaf.

Ex $y : \mathcal{O}(X) \hookrightarrow Set^{\mathcal{O}(X)^{op}}$

In this case, $y(U)(V) = \begin{cases} * & \text{if } V \subset U \\ \emptyset & \text{otherwise.} \end{cases}$

Let $(V_i \hookrightarrow V)_i$ be an open cover. Then, if $\overset{\bigcup_i V_i}{\overset{\shortparallel}{V}} \subset U$,

$y(U)(V_i) = *$ the terminal set, $\forall i$ (and also $y(U)(V_{ij}) = *$) $\Rightarrow$

$$\underset{*}{\underset{\shortparallel}{y(U)(V)}} \rightleftarrows \lim \Big[\prod_i y(U)(V_i) \rightrightarrows \prod_{i,j} y(U)(V_{ij}) \Big] \cong *$$ so is an iso.

If $V \not\subset U$, then this becomes

$\emptyset \longrightarrow \emptyset$ which is also an iso, so $y(U)$ is a sheaf.

Ex Let A be a set. Consider the functor

$$\Delta_A : \mathcal{O}(X)^{op} \longrightarrow Set$$ with constant value A.

Δ_A cannot be a sheaf unless $A \cong *$, since we need $\Delta_A(\emptyset) \cong *$.

Can we find another functor $\hat{\Delta}_A : \mathcal{O}(X)^{op} \longrightarrow Set$ s.t.
$\hat{\Delta}_A(\emptyset) = *$ and $\Delta_A(U) = A$ $\forall$ $U \neq \emptyset$? (and $\hat{\Delta}_A(\emptyset \xrightarrow{!} U) : \hat{\Delta}_A(U) = A \xrightarrow{!} * = \hat{\Delta}_A(\emptyset)$)

Let $X = \{0, 1\}$ disc. Suppose $A \ncong *$. Let $a_0 \in \hat{\Delta}_A(\{0\}) = A$

$$\overset{\neq}{a_1} \in \hat{\Delta}_A(\{1\}) = A$$

Restriction maps $\hat{\Delta}_A(X) = A \xrightarrow[id_A]{} A = \hat{\Delta}_A(\{0\})$ and similarly for $\{1\}$.
$\nexists$ $a \in \hat{\Delta}_A(X)$ s.t. $a|_{\{1\}} = a = a_0$ so $\hat{\Delta}_A$ is not a sheaf.
and $a|_{\{0\}} = a = a_1$

Note $C(\cdot, A_{disc})$ is a sheaf $\forall$ A and $C(U, A_{disc}) \cong Hom(\pi_0(U), A)$.

<u>Def</u> Let $f : X \longrightarrow Y$ be a continuous map.

$$\rightsquigarrow \quad f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$$
$$U \longmapsto f^{-1}(U).$$

Given a presheaf $F \in Set^{\mathcal{O}(X)^{op}}$, denote by $f_* F$ the composite

$$\mathcal{O}(Y)^{op} \xrightarrow{(f^{-1})^{op}} \mathcal{O}(X)^{op} \xrightarrow{F} Set,$$

i.e. $f_* F(U) = F(f^{-1}(U))$.

$f_* F$ is called the <u>direct image</u> of F, $\rightsquigarrow$

$$f_* : Set^{\mathcal{O}(X)^{op}} \longrightarrow Set^{\mathcal{O}(Y)^{op}} \quad \text{(functor)},$$

f_* is called the <u>direct image functor</u>.

<u>Prop</u> If $F \in Sh(X)$, $f_* F \in Sh(Y)$.

<u>Pf</u> f^{-1} preserves open covers. $\quad Sh(*) \overset{\Delta x}{\simeq} Set$, $\quad C(\cdot, X_{disc}) \,/\, F \longmapsto F(*)$, $\quad \hookleftarrow\!\!| \; X$

$Sh(*) \hookrightarrow Set^{2^{op}}$, $2 = \phi \overset{!}{\longrightarrow} *$

<u>Ex</u>: Let $x \in X$ considered as a map $x : * \longrightarrow X$.

$$\rightsquigarrow \quad x_* : Sh(*) \simeq Set \longrightarrow Sh(X)$$
$$(C(\cdot, A_{disc})) \leftarrow\!\!| \; A \longmapsto x_* A.$$

$x_* A$ is usually denoted by $Sky_x(A)$ and is called the <u>skyscraper sheaf of A concentrated at x</u> since

$$Sky_x(A)(U) = \begin{cases} A & \text{if } x \in U \\ * & \text{if } x \notin U \end{cases} \quad \Big| \quad \left(\text{since } \begin{array}{l} x^{-1}(U) = * \text{ if } x \in U \\ \phantom{x^{-1}(U)} = \phi \text{ otherwise} \end{array} \right)$$