Presheaf Categories (Lecture 1)

<u>Def</u> If $\mathcal{C}$ & $\mathcal{D}$ are categories, denote by $\mathcal{D}^{\mathcal{C}}$

the category: <u>objects</u> $F : \mathcal{C} \to \mathcal{D}$ functors

<u>arrows</u>
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd}$$
natural transformations

For any (essentially) small category $\mathcal{C}$, the category $\mathrm{Set}^{\mathcal{C}^{op}}$ is called the category of <u>presheaves</u> on $\mathcal{C}$, and is a <u>topos</u>.

Examples:

0) Set $(\mathcal{C} = *)$

1) M-Set. Regard M as a one object category $\begin{tikzcd} * \arrow[loop above, "M"] \end{tikzcd}$

$\exists$ a functor $\mathrm{Set}^{M} \xrightarrow{\Theta} M\text{-Set}$

<u>objects</u>: $F : M \to \mathrm{Set} \mapsto (M \circlearrowright F(*))$ | if $x \in F(*)$, $m \cdot x = F(m)(x)$

<u>arrows</u>:
$$\begin{tikzcd} M \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathrm{Set} \end{tikzcd} \rightsquigarrow f_\alpha = \alpha(*) : F(*) \to G(*)$$

$$\rightsquigarrow \quad \begin{tikzcd} F(*) \arrow[r, "\alpha(*) = f_\alpha"] \arrow[d, "F(m)"'] \arrow[dr, phantom, "\circlearrowright"] & G(*) \arrow[d, "G(m)"] \\ F(*) \arrow[r, "\alpha(*)"'] & G(*) \end{tikzcd} \quad \forall m \quad \Leftrightarrow \quad \forall x \in F(*) \quad m \cdot f_\alpha(x) = f_\alpha(m \cdot x).$$

$$\Upsilon : M\text{-Set} \longrightarrow \mathrm{Set}^M$$

objects

$$\rho : M \times X \to X \longmapsto \Upsilon(\rho) : * \mapsto X$$

$$\Upsilon(\rho)(m) = \rho(m, \cdot) : X \to X.$$

arrows $M \overset{\rho}{\curvearrowright} X \xrightarrow{f} M \overset{\lambda}{\curvearrowright} Y$, $\Upsilon(f)(*) = f$.

$\Theta\Upsilon = id_{\mathrm{Set}^M}$, $\Upsilon\Theta = id_{M\text{-Set}}$.

So $\underset{(\text{left})}{M\text{-Set}} \cong \mathrm{Set}^M = \mathrm{Set}^{(M^{op})^{op}}$.

1') $\underset{(\text{left})}{G\text{-Set}} \cong \mathrm{Set}^G \cong \mathrm{Set}^{G^{op}}$ since $G \cong G^{op}$

$\mathrm{Set}^{G^{op}} = (\text{right})\ \mathrm{Set}\text{-}G$

2) Let $\mathcal{C}$ be a small category, and X a set.

Def Define a (left)-action of $\mathcal{C}$ on X as the following data

a moment map $\mu : X \longrightarrow \mathcal{C}_0$.

Consider the pullback $\{(f, x) \mid f : \mu(x) \to t(f)\}$

$$\begin{tikzcd} \mathcal{C}_1 \times_{\mathcal{C}_0} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\mu"] \\ \mathcal{C}_1 \arrow[r, "s"'] & \mathcal{C}_0 \end{tikzcd}$$

(source)

An action map $\rho : \mathcal{C}_1 \times_{\mathcal{C}_0} X \longrightarrow X$

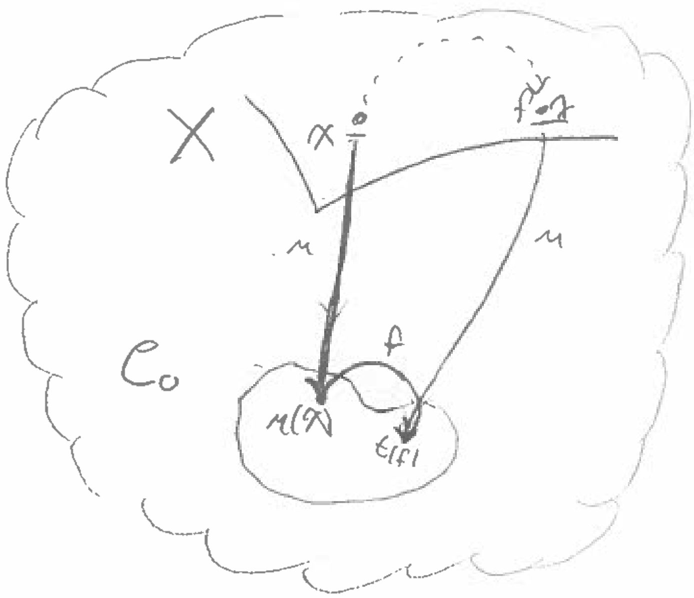
$$(f, x) \longmapsto f \cdot x$$

s.t.

i) $(gf)\cdot x = g\cdot(f\cdot x)$ whenever this makes sense

ii) $1_{\mu(x)}\cdot x = x \quad \forall\, x$

iii) $\mu(f\cdot x) = t(f)$ (target)



Call a set X with a $\mathcal{C}$-action a (left) $\mathcal{C}$-Set.

A morphism $\mathcal{C} \curvearrowright X \longrightarrow \mathcal{C} \curvearrowright Y$ is a function

i)
$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\mu_X"'] & \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dl, "\mu_Y"] \\ & \mathcal{C}_0 & \end{tikzcd}$$

s.t. ii) $\varphi(f\cdot x) = f\cdot\varphi(x)$

$\boxed{\text{Homework: Prove } \mathcal{C}\text{-Set} \cong \mathrm{Set}^{\mathcal{C}}.}$

Other examples:

3) Graph = objects = (directed) graphs

arrows = functions F between the set of vertices & the set of edges s.t.

if $A \xrightarrow{e} B$ is an edge

$F(A) \xrightarrow{F(e)} F(B)$

Let $\mathcal{J} =$
$$\begin{tikzcd} z \arrow[loop above, "id"] & y \arrow[l, "g"'] \arrow[loop above, "id"] \\ x \arrow[u, "f"] \arrow[loop below, "id"] & \end{tikzcd}$$
category. Then Graph $\cong \mathrm{Set}^{\mathcal{J}^{op}}$.

4) $\mathrm{Set}^{\Delta^{op}}$ = simplicial sets.

The Yoneda Lemma

<u>Def</u> Let $C \in \mathcal{C}_0$. Define

$$y(C) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$
$$D \longmapsto \mathrm{Hom}(D, C)$$

$\boxed{y(C) \text{ is a functor by associativity}}$

$$D \xrightarrow{g} E \mapsto \mathrm{Hom}(E, C) \xrightarrow{y(C)^{*}(g)} \mathrm{Hom}(D, C)$$
$$h : E \to C \longmapsto D \xrightarrow{g} E \xrightarrow{h} C$$
$$h \longmapsto hg$$

$y(C)$ is called the presheaf <u>represented</u> by C, and is called <u>representable</u>.

The assignment y assembles into a functor

$$y : \mathcal{C} \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$$

$$C \xrightarrow{f} D \longmapsto (y(C)(E) \xrightarrow{\tilde{f}_E = f^{*}} y(D)(E)$$
$$E \xrightarrow{g} C \longmapsto E \xrightarrow{g} C \xrightarrow{f} D$$
$$g \longmapsto fg$$

$\boxed{y \text{ is a functor by associativity}}$

<u>Yoneda Lemma</u>:

For every presheaf F, and $C \in \mathcal{C}_0$, there is a natural bijection

$$\mathrm{Hom}(y(C), F) \xrightarrow{\sim} F(C).$$

<u>Pf</u> Let $\lambda : y(C) \Longrightarrow F$.

Suppose we want to describe the component

$$\begin{aligned} \lambda_D : y(C)(D) = \mathrm{Hom}(D,C) &\longrightarrow F(D) \\ f &\longmapsto \lambda_D(f). \end{aligned}$$

Since λ is natural, the following diagram commutes:

$$\begin{tikzcd}
id_C \arrow[rrr, mapsto, bend left] \arrow[dddd, mapsto, bend right=60] & & & \lambda_C(id_C) \in F(C) \arrow[dd, mapsto] \\
y(C)(C) \arrow[r, "\lambda_C"] \arrow[dd, "y(C)(f)"'] & F(C) \arrow[dd, "F(f)"] & & \\
 & & & F(f)(\lambda_C(id_C)) \\
y(C)(D) \arrow[r, "\lambda_D"] & F(D) & & \rotatebox{60}{$=$} \\
f \arrow[rrr, mapsto] & & & \lambda_D(f) \quad (*)
\end{tikzcd}$$

Let

$$\begin{aligned} \Phi_C : \mathrm{Hom}(y(C), F) &\longrightarrow F(C). \\ \lambda &\longmapsto \lambda_C(id_C) \end{aligned}$$

Define

$$\begin{aligned} \Psi_C : F(C) &\longrightarrow \mathrm{Hom}(y(C), F) \\ x &\longmapsto \Psi(x) : y(C) \Longrightarrow F \end{aligned}$$

$$\begin{aligned} \Psi(x)_D : \mathrm{Hom}(D,C) &\longrightarrow F(D) \\ f &\longmapsto F(f)(x). \end{aligned}$$

<u>Naturality</u>: let $g : E \to D$

$$\begin{tikzcd}
f \arrow[rrr, mapsto, bend left] \arrow[dddd, mapsto, bend right=60] & & & F(f)(x) \arrow[dd, mapsto] \\
y(C)(D) \arrow[r, "\Psi(x)_D"] \arrow[dd, "y(C)(g)"'] & F(D) \arrow[dd, "F(g)"] & & \\
 & & & F(g)(F(f)(x)) \\
y(C)(E) \arrow[r, "\Psi(x)_E"'] & F(E) & & \rotatebox{60}{$=$} \quad \checkmark \\
fg \arrow[rrr, mapsto] & & & F(fg)(x)
\end{tikzcd}$$

By $(*)$ $\psi_c \varphi_c = id$.

and $\varphi_c \psi_c(x) = F(id)(x) = x$ $\square$

In fact $(\varphi_C : \mathrm{Hom}(y(C), F) \longrightarrow F(C))$ are the components of a natural isomorphism

$$\mathrm{Hom}(y(\cdot), F) \overset{\sim}{\Longrightarrow} F.$$

<u>Corollary:</u>
The functor $y : \mathcal{C} \longrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$ is full and faithful.

<u>Pf</u>: WTS $\forall$ C & D the map

$$\mathrm{Hom}(C,D) \xrightarrow{y_{C,D}} \mathrm{Hom}(y(C), y(D)) \text{ is an isomorphism.}$$

Consider the composite

$$\begin{tikzcd}
f \arrow[r, mapsto] & y(f) \arrow[rr, bend left, mapsto] & & y(f)_C(id_C) = f \\
\mathrm{Hom}(C,D) \arrow[r, "y_{C,D}"] \arrow[dr, "id"'] & \mathrm{Hom}(y(C), y(D)) \arrow[d, "\varphi_C", "\wr"'] & & \\
& y(D)(C) = \mathrm{Hom}(C,D) & &
\end{tikzcd}$$

$\Rightarrow$ $y_{C,D}$ is an iso.

Because of this, we often write C instead of $y(C)$.

<u>Rmk</u> If $\mathcal{C} = \Delta$, the Yoneda lemma recovers the well known fact that if X is a simplicial set,

$$X_n \cong \mathrm{Hom}(\Delta[n], X).$$