Weekly Homework 8

Instructor: David Carchedi
Topos Theory

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Problem 1. Subobject classifiers

Read the section on subobject classifiers in the lecture notes entitled "Lecture 8" posted on the website. In these, it is proven that any presheaf category $\mathbf{Set}^{\mathscr{C}^{op}}$ has a subobject classifier.

(a) Prove that if J is a Grothendieck topology on $\mathscr{C}$, then $\mathscr{E} = \mathbf{Sh}_J(\mathscr{C})$ has a subobject classifier Ω_J.

(b) Prove that if $\mathscr{D}$ is any category with a subobject classifier

$$t : T \to \Omega,$$

then T must be the terminal object.

Problem 2. Lawvere-Tierney topologies

(a) Show that if Ω is a subobject classifier of a topos $\mathscr{E}$, that taking intersections of subobjects induces a morphism

$$\wedge : \Omega \times \Omega \to \Omega.$$

Definition 1. A **Lawvere-Tierney topology** on a topos $\mathscr{E}$, with subobject classifier

$$t : 1 \to \Omega$$

is an idempotent J of Ω (i.e. $J^2 = J$) such that the following two diagrams commute

$$\begin{array}{ccc} 1 & \xrightarrow{t} & \Omega \\ & {\scriptstyle t}\searrow & \downarrow{\scriptstyle J} \\ & & \Omega \end{array}$$

$$\begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ {\scriptstyle J \times J}\downarrow & & \downarrow{\scriptstyle J} \\ \Omega \times \Omega & \xrightarrow[\wedge]{} & \Omega. \end{array}$$

(b) Prove that for any small category $\mathscr{C}$, there is a bijection between Grothendieck topologies on $\mathscr{C}$ and Lawvere-Tierney topologies on $\mathbf{Set}^{\mathscr{C}^{op}}$.
Hint: See HW7. (Also, this exercise may be helpful in proving HW7.)

Problem 3. Slices of topoi

Let $(\mathscr{C}, J)$ be a Grothendieck site and let $F \in \mathbf{Sh}_J(\mathscr{C})$ be a sheaf. Describe a Grothendieck topology $J|_F$ on

$$\int_{\mathscr{C}} F,$$

such that

$$\mathbf{Sh}_J(\mathscr{C})/F \simeq \mathbf{Sh}_{J|_F}\left(\int_{\mathscr{C}} F\right).$$

This proves that for any Grothendieck topos $\mathscr{E}$, and any object $E \in \mathscr{E}$, $\mathscr{E}/E$ is a Grothendieck topos.
See HW1, Problem $2\ f)$**.**