Weekly Homework 7

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Topos Theory

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Problem 1.   Universal Closure Operations and Left Exact Localizations

Definition 1. A **universal closure operations** on $\mathbf{Set}^{\mathscr{C}^{op}}$ is an assignment to each presheaf X an operation

$$\overline{(\,\cdot\,)} : \mathbf{Sub}\,(X) \to \mathbf{Sub}\,(X)$$

such that

  i) $A \le \bar{A}$

  ii) $\bar{A} = \bar{\bar{A}}$

  iii) $A \le B \Rightarrow \bar{A} \le \bar{B}$

  iv) For $f : Y \to X$ and $A \in \mathbf{Sub}\,(X)$, $f^*\left(\bar{A}\right) = \overline{f^*\,(A)}$.

  Prove that the following data are equivalent for a small category $\mathscr{C}$:

A) A Grothendieck topology on $\mathscr{C}$,

B) (isomorphism classes of) left exact localizations of $\mathbf{Set}^{\mathscr{C}^{op}}$ − i.e. full subcategories

$$i : \mathcal{E} \hookrightarrow \mathbf{Set}^{\mathscr{C}^{op}},$$

  such that i admits a left exact left adjoint L.

C) A universal closure operation on $\mathbf{Set}^{\mathscr{C}^{op}}$