# Weekly Homework 2

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Topos Theory

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## Problem 1. Examples of Presheaves

(a) Let $X, Y$, and $Z$ be topological spaces. Consider the presheaves $C(\cdot, Y)$ and $C(\cdot, Z)$ on $X$ of continuous functions into $Y$ and $Z$ respectively, and consider their coproduct in the category $\mathbf{Set}^{\mathscr{O}(X)^{op}}$,

$$C(\cdot, Y) \coprod C(\cdot, Z).$$

Show that it is not a sheaf (in general). Show that the sheaf $C(\cdot, Y \coprod Z)$ is a sheaf, and that it is the coproduct of $C(\cdot, Y)$ and $C(\cdot, Z)$ in the category $\mathbf{Sh}(X)$.

(b) Consider the presheaf $K_0$ on the 2-sphere $S^2$, which assigns each open subset $U$ the set of isomorphism classes of finite dimensional vector bundles over $U$. Show that it is not separated.

(c) Let $\pi : V \to X$ be a vector bundle. Show that the assignment to each open subset $U$ the set of vector bundle automorphisms of $V|_U$ (covering the identity of $U$) assembles naturally into a presheaf $\mathbf{Aut}(V)$ on $X$. Prove $\mathbf{Aut}(V)$ is a sheaf, or give a counter-example.

## Problem 2. Adjunctions and the Free Co-limit Co-completion

(a) Show that if

$$\mathscr{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathscr{D},$$

is an adjunction with $F$ left adjoint to $G$ (written $F \dashv G$), then the co-unit $\varepsilon : FG \Rightarrow id_{\mathscr{D}}$ is an isomorphism if and only if the *right* adjoint $G$ is full and faithful.

(b) Show every adjunction induces an equivalence between, on one hand, the full subcategory of $\mathscr{C}$ on which the unit is an isomorphism, and on the other hand, the full subcategory of $\mathscr{D}$ on which the co-unit is an isomorphism.

(c) Let $\mathscr{C}$ and $\mathscr{D}$ be categories with $\mathscr{C}$ small. Consider the functor

$$y^* : \mathscr{D}^{\mathbf{Set}^{\mathscr{C}^{op}}} \to \mathscr{D}^{\mathscr{C}}$$

given by precomposition with the Yoneda embedding $y$. Show that if $\mathscr{D}$ is cocomplete, then $y^*$ has a left adjoint, $\mathbf{Lan}_y(\,\cdot\,)$. Show that unit is always an isomorphism, and identify the essential image of $\mathbf{Lan}_y(\,\cdot\,)$ as those functors

$$\theta : \mathbf{Set}^{\mathscr{C}^{op}} \to \mathscr{D}$$

which preserve small colimits, denoted by $\mathscr{D}^{\mathbf{Set}^{\mathscr{C}^{op}}}_{cocont.}$. Combine this with $(a)$ to show that precomposition with $y$ induces an equivalence of categories

$$\mathscr{D}^{\mathbf{Set}^{\mathscr{C}^{op}}}_{cocont.} \xrightarrow{\sim} \mathscr{D}^{\mathscr{C}}.$$

In particular, this shows that any functor

$$F : \mathscr{C} \to \mathscr{D}$$

extends to a colimit preserving functor

$$\tilde{F} : \mathbf{Set}^{\mathscr{C}^{op}} \to \mathscr{D},$$

unique up to natural isomorphism, such that $\tilde{F}$ agrees with $F$ on representables.

(d) Use the Yoneda lemma to show that in the situation above, if $F : \mathscr{C} \to \mathscr{D}$, that $\mathbf{Lan}_y(F)$ has a right adjoint $R_F$, and give and explicit formula for $R_F$ on objects. Show that if $\mathscr{C} = \Delta$ is the simplex category and

$$\Delta^\bullet : \Delta \to Top$$

is the standard cosimplicial space

$$[n] \mapsto \Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,|\, t_i \geq 0,\, t_i = 1\},$$

then adjunction $\mathbf{Lan}_y(\Delta^\bullet) \dashv R_{\Delta^\bullet}$ is the standard geometric realization functor / singular nerve adjunction. If instead, one considers the full and faithful inclusion

$$i : \Delta \hookrightarrow \mathbf{Cat}$$

of $\Delta$ into the category of small categories, show that the right adjoint to $\mathbf{Lan}_y(i)$ is the functor

$$N : \mathbf{Cat} \to \mathbf{Set}^{\Delta^{op}}$$

assigning a small category its nerve.