Fixed points of isometries on weakly compact convex sets

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Received 22 October 2001
Submitted by W.A. Kirk

Abstract
In this paper, we prove that every isometry from a nonempty weakly compact convex set $K$ into itself fixes a point in the Chebyshev center of $K$, provided $K$ satisfies the hereditary fixed point property for isometries. In particular, all isometries from a nonempty bounded closed convex subset of a uniformly convex Banach space into itself have the Chebyshev center as a common fixed point.

Keywords: Weakly compact convex set; Isometry; Fixed point; Uniformly convex Banach space

1. Introduction
Let $X$ be a Banach space and $S$ a bounded subset of $X$. For $x \in X$, we write

$$R(x, S) = \sup \{ \|x - y\| : y \in S \},$$

and we denote the diameter of $S$ by $\text{diam } S$, i.e., $\text{diam } S = \sup \{ \|y - z\| : y, z \in S \}$. For $r \geq 0$, $B(x, r)$ denotes the closed ball with radius $r$ centered at $x$. $\overline{\text{co}}(S)$ denotes the closure of the convex hull of $S$.
Let $K$ be a weakly compact convex nonempty subset of $X$. Let $R_K = \inf \{ R(x, K) : x \in K \}$. The set $\{ x \in K : R(x, K) = R_K \}$ and the number $R_K$ are called, respectively, the Chebyshev center and the Chebyshev radius of $K$. The subset $K$ of $X$ is said to have normal structure if every convex subset $C$ of $K$ with more than one point has a nondiametral point, i.e., a point $x_0 \in C$, such that

$$R(x_0, C) < \text{diam } C.$$ 

A mapping $T : K \rightarrow K$ is called isometry if $\| T x - T y \| = \| x - y \|$ for all $x, y$ in $K$.

The following proposition is known; we give its proof for completeness.

**Proposition 1.** The Chebyshev center $C$ of $K$ is nonempty. If $T : K \rightarrow K$ is an isometry from $K$ onto $K$, then $T(C) = C$.

**Proof.** We have

$$C = \{ x \in K : R(x, K) = R_K \} = \bigcap_{n \in \mathbb{N}} C_n,$$

where each $C_n = \{ x \in K : R(x, K) \leq R_K + 1/n \}$ is nonempty closed convex. Since $K$ is weakly compact and $\{ C_n \}$ is decreasing, $C \neq \emptyset$.

Let $x \in C$. Since $T$ is a surjective isometry, one has

$$R(T x, K) = R(T x, T K) = R(x, K) = R_K$$

and hence $T x \in C$. So $T(C) \subset C$.

On the other hand, applying the above argument to $T^{-1}$, we get that $x \in C$ implies $y = T^{-1}x \in C$. So $x = Ty \in T(C)$, showing that $C \subset T(C)$. $\square$

**Question 1.** In Proposition 1, if $T$ is not surjective, does one still have $T(C) \subset C$?

We will see later that the answer to the above question is positive under certain assumptions.

2. Isometries in uniformly convex spaces

Throughout this section we denote by $X$ a uniformly convex Banach space and by $K$ a nonempty bounded closed and convex subset of $X$. Since every uniformly convex Banach space is reflexive, the set $K$ is weakly compact.

An equivalent definition of uniform convexity is: for every $\epsilon > 0$ and $r > 0$, there exists $\delta(\epsilon) > 0$ such that whenever $x, y, z \in X$, $\| x - z \| \leq r$, $\| y - z \| \leq r$ and $\| x - y \| \geq \epsilon$, one has

$$\left\| \frac{x + y}{2} - z \right\| \leq r - \delta(\epsilon).$$

**Lemma 1.** Let $K$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$. The Chebyshev center $C$ of $K$ is a singleton, say $k_0$. 
Proof. That $C \neq \emptyset$ follows from Proposition 1. Suppose on the contrary that $x_1, x_2 \in C$, and let $\|x_1 - x_2\| = \epsilon > 0$. Let $r = R_K$. For every $z \in K$, we have $\|x_1 - z\| \leq r$ and $\|x_2 - z\| \leq r$. By the definition of uniform convexity, there exists a $\delta(\epsilon) > 0$ such that
\[
\left\| \frac{x_1 + x_2}{2} - z \right\| < r - \delta(\epsilon).
\] (1)
By the definition of $R_K$, there exists a $z$ in $K$ such that
\[
\left\| \frac{x_1 + x_2}{2} - z \right\| > r - \delta(\epsilon),
\] (2)
which contradicts (1).
\[\Box\]

Proposition 2. Let $K$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$. If $T$ is an isometry, $T : K \rightarrow K$, then $T(K)$ is also nonempty bounded closed convex.

Proof. It is well known that any isometry in a uniformly convex Banach space is affine [2]. So the convexity of $T(K)$ is obvious.

If $\{T x_n\}$ is a sequence in $T(K)$, which converges to $y$, then $\{T x_n\}$ is a Cauchy sequence. By isometry $\{x_n\}$ is also Cauchy and so converges to a point $x$ of the complete set $K$. Therefore $y = Tx \in T(K)$ and $T(K)$ is closed. \[\Box\]

Lemma 2. The Chebyshev center $k$ of $K$ is a fixed point for all isometries from $K$ onto $K$.

Proof. By Lemma 1, the Chebyshev center is unique. By Proposition 1, it is a fixed point of every isometry from $K$ onto $K$. \[\Box\]

Lemma 3. Let $T : K \rightarrow K$ be an isometry. Then there exists a $k$ in $K$ such that $Tk = k$.

Proof. We have $T^2(K) \subseteq T(K) \subseteq K$ and inductively $T^{n+1}(K) \subseteq T^n(K)$. Consider the set $L := \bigcap\{T^n(K) : n \in \mathbb{N}\}$. Since $T^n(K)$ is weakly compact for every $n \in \mathbb{N}$, it follows that $L \neq \emptyset$. Using the fact that $T$ is one-to-one, it is not difficult to show that $T(L) = L$, and so our conclusion follows from Lemma 2. \[\Box\]

Lemma 4. Let $T : K \rightarrow K$ be an isometry and consider the set $S = \overline{\text{co}}(T^n x_0 : n \in \mathbb{N})$, where $x_0$ is an arbitrary point of $K$. Then $T$ has a fixed point in $S$.

Proof. Obviously since $T$ is affine and continuous, $T(S) \subseteq S$, and the proof follows from Lemma 3. \[\Box\]

Lemma 5. Let $K$ be as in Lemma 1 and $T : K \rightarrow K$ an isometry. Then the Chebyshev radius $R_T(K)$ of $T(K)$ is equal to $R_K$ and the Chebyshev center of $T(K)$ coincides with the point $Tk_0$. 

Proof. For the point \( T x_0 \in T(K) \) we have
\[
\| T k_0 - Tx \| = \| k_0 - x \| \leq R_K, \quad \forall Tx \in T(K),
\]
which implies that \( R_{T(K)} \leq R_K \).

Suppose for a contradiction that \( R_{T(K)} < R_K \) and let \( k_0' \) be the Chebyshev center of \( T(K) \). Then there is a point \( y \in K \) such that \( Ty = k_0' \). Now for every \( x \in K \) we have
\[
\| x - y \| = \| Tx - Ty \| \leq R_{T(K)} < R_K,
\]
which contradicts the definition of \( R_K \).

Hence \( R_{T(K)} = R_K \) and (3) implies that \( T(k_0) \) is the Chebyshev center of \( T(K) \). \( \square \)

Now we can prove our result:

Theorem 1. Let \( K \) be a nonempty bounded closed and convex subset of a uniformly convex Banach space \( X \). Then the Chebyshev center \( k_0 \) of \( K \) is a fixed point for every isometry \( T : K \to K \).

Proof. Suppose on the contrary that \( T k_0 \neq k_0 \) and consider the set \( S = \mathcal{C}[T^n k_0 : n \in \mathbb{N}] \).
From Lemma 4 we have that there is a point \( z \in S \) with \( Tz = z \). Obviously \( z \in T^n(K) \), \( \forall n \in \mathbb{N} \), and because \( z \) is not the Chebyshev center of \( K \) there exists a point \( e \in K \) such that
\[
\| z - e \| = R_K + t
\]
for some \( t > 0 \). Since \( z \in S \), it follows that there exist nondecreasing sequence \( n_1, n_2, \ldots, n_k \in \mathbb{N} \) with
\[
\left| \sum_{i=1}^{k} \lambda_i T^{n_i} k_0 - z \right| < \frac{t}{2}
\]
where \( \sum_{i=1}^{k} \lambda_i = 1, \lambda_i > 0, \forall i = 1, 2, \ldots, k \). From (4) and (5) it follows that
\[
\left| \sum_{i=1}^{k} \lambda_i T^{n_i} k_0 - T^{n_k} e \right| \geq \| z - T^{n_k} e \| - \left| \sum_{i=1}^{k-1} \lambda_i T^{n_i} k_0 \right|
\]
\[
\geq R_K + t - \frac{t}{2} = R_K + \frac{t}{2}.
\]
On the other hand, since \( T^{n_k} k_0, T^{n_i} e \in T^{n_i}(K) \), \( \forall i = 1, 2, \ldots, k \), Lemma 5 implies that \( \| T^{n_k} k_0 - T^{n_i} e \| \leq R_K \), and we have
\[
\left| \sum_{i=1}^{k} \lambda_i T^{n_i} k_0 - T^{n_k} e \right| \leq \sum_{i=1}^{k} \lambda_i \| T^{n_i} k_0 - T^{n_i} e \| \leq R_K.
\]
Now (7) contradicts (6) and we obtain the result. \( \square \)
3. Isometries on weakly compact convex sets

Let \( K \) be a weakly compact convex subset of a Banach space. Let \( \{B_\alpha: \alpha \in \Lambda\} \) be a decreasing net of nonempty subsets of \( K \) and for each \( x \in K \) let

\[
  r(x, \{B_\alpha: \alpha \in \Lambda\}) = \lim_{\alpha} R(x, B_\alpha)
\]

and

\[
  r = r(K, \{B_\alpha: \alpha \in \Lambda\}) = \inf \{r(x, \{B_\alpha: \alpha \in \Lambda\}): x \in K\}.
\]

The number \( r \) and the set \( \{x \in K: r(x, \{B_\alpha: \alpha \in \Lambda\}) = r\} \), which is nonempty closed convex, are called, respectively, the asymptotic radius and asymptotic center of \( \{B_\alpha: \alpha \in \Lambda\} \) with respect to (w.r.t.) \( K \).

The following can be found in [6].

**Proposition 3.** If \( T : K \to K \) is nonexpansive, i.e., \( \|Tx - Ty\| \leq \|x - y\| \), \( \forall x, y \in K \), then the asymptotic center of the sequence of sets \( T^n(K) \) is \( T \)-invariant.

**Proof.** Let \( C \) be the asymptotic center of \( \{T^n(K)\} \) w.r.t. \( K \) and \( r \) be its asymptotic radius. Let \( x \in C \). For each \( y \in K \) we have

\[
  \|T^n y - Tx\| \leq \|T^{n-1} y - x\|
\]

which implies

\[
  R(Tx, T^n(K)) \leq R(x, T^{n-1}(K))
\]

and hence

\[
  r(Tx, \{T^n(K)\}) \leq r(x, \{T^n(K)\}).
\]

Therefore \( Tx \in C \). \( \square \)

We say a weakly compact convex nonempty subset \( K \) of a Banach space have fixed point property (f.p.p.) if every isometry of \( K \) into \( K \) has a fixed point. \( K \) is said to have hereditary f.p.p. if every closed convex nonempty subset of \( K \) has the f.p.p. It is well known (see Kirk [5]) that every weakly compact convex nonempty subset of a Banach space with normal structure has the hereditary f.p.p. Also from a result of Maurey (see [3, Theorem F]), every closed convex bounded nonempty subset of a superreflexive space has hereditary f.p.p. The following theorem is more general than Theorem 1. Since the proofs in Section 2 are essentially self-contained, we think that Theorem 1 warrants separate treatment.

**Theorem 2.** Let \( K \) be a weakly compact convex nonempty subset of a Banach space, and assume that \( K \) has the hereditary f.p.p. Let \( T : K \to K \) be an isometry. Then \( T \) has a fixed point in the Chebyshev center of \( K \).
Proof. Let $C$ and $r$ be the asymptotic center and the asymptotic radius, respectively, of the sequence $T^n(K)$ w.r.t. $K$. Since $C$ is $T$-invariant, $T$ has a fixed point $c$ in $C$. Since $T$ is an isometry and $T(c) = c$, we have

$$ R(c, T^n(K)) = R(Tc, T^n(K)) = R(c, T^{n-1}(K)), \quad n = 1, 2, \ldots, $$

from which it follows that $r = R(c, K)$. For any $x \in K$, we have

$$ R(x, K) \geq \lim_{n} R(x, T^n(K)) \geq r = R(c, K). $$

Thus $c$ is in the Chebyshev center of $K$. \qed

The following simple example shows that the isometry $T$ in the theorem above need not fix all points in the Chebyshev center.

Example 1. Let $X$ be $\mathbb{R}^2$ with the sup norm. Let $K$ be the right half of the unit ball of $X$. The Chebyshev center of $K$ is $\{(x, 0) : 0 \leq x \leq 1\}$ and the isometry $T(x, y) = (1 - x, y)$ fixes only $(1/2, 0)$ in the Chebyshev center.

Corollary 1. Let $K$ be a weakly compact convex nonempty subset of a Banach space, and assume that $K$ has normal structure. Let $T : K \to K$ be an isometry. Then $T$ has a fixed point in the Chebyshev center of $K$.

Corollary 2. Let $K$ be a closed convex bounded nonempty subset of a superreflexive Banach space, and $T : K \to K$ be an isometry. Then $T$ has a fixed point in the Chebyshev center of $K$.

Corollary 3. Let $K$ be a weakly compact convex nonempty subset of a Banach space whose norm is uniformly convex in every direction. Then every isometry $T : K \to K$ has the Chebyshev center of $K$ as a fixed point.

Proof. The Chebyshev center of $K$ is unique [4]. \qed

Problem 1. Let $K$ be a weakly compact convex nonempty subset of a Banach space, and assume that $K$ has normal structure. Brodskii–Milman [1] proved that there is a point that is fixed by every surjective isometry of $K$. Does $K$ have a point that is fixed by every isometry from $K$ into $K$?

References

