Addendum to ”Fixed Point Theorems for Uniformly Lipschitzian Mappings in \( L^p \) Spaces”

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In the proof of Lemma 2 in [1], one minor step was missing in the process of proving that \( x(0+) \) is the solution of

\[
(r - 1)x^r + rx^{r-1} - 1 = 0
\]

in the interval \( 0 < x < 1 \).

We have

\[
\lambda \mu \frac{1 - \lambda^{r-1}}{1 - \lambda} x(\mu)^r - \mu + r \lambda^{r-1} \mu x(\mu)^{r-1} - \left( \frac{r}{2} \right) \lambda^{r-2} \mu^2 x(\mu)^{r-2} + \ldots
\]

\[
= \lambda \mu \frac{1 - \lambda^{r-1}}{1 - \lambda} x(\mu)^r - \mu + r \lambda^{r-1} \mu x(\mu)^{r-1} + \lambda^r x(\mu)^{r-1} \alpha(\alpha),
\]

where \( \alpha = \frac{\mu}{\mu + x(\mu)} \). Thus \( \alpha(\alpha)/\mu \to 0 \) as \( \mu \to 0 \) if one can prove that \( x(\mu) \) is bounded away from 0. To this end, we shall prove that \( x(\mu) \geq 1/4 \) for \( 0 < \mu \leq 1/5 \) (and hence for all \( \mu \in (0, 1/2] \) since \( x(\mu) \) is strictly increasing). For \( 0 < \mu \leq 1/5 \), we have \( \mu/\lambda \leq 1/4 \); thus it suffices to show that \( h(1/4) < 0 \), or equivalently,

\[
H(\mu) = \lambda - \mu 4^r - (\lambda - 4\mu)^r < 0.
\]

We have \( H(0) = 0 \) and \( H'(\mu) = 5r(\lambda - 4\mu)^{r-1} - 4r - 1 < 5r - 4^r - 1 \). Simple calculus shows that \( 5r - 4^r - 1 < 0 \). Thus \( H(\mu) \) is strictly decreasing and \( H(\mu) < 0 \).

Also, the proof of Lemma 4 implies that \( f(\mu) = \frac{\alpha(\mu)}{\lambda^r} \) is strictly decreasing in \( \mu \); thus we have

\[
2^{2-p} = f(1/2) < f(0+) = \frac{1 + \alpha^{p-1}}{(1 + \alpha)^{p-1}}.
\]

Therefore we should have an improved bound for \( L \) in Theorem 2, namely, \( L > (1 + 2^{2-p})^{1/p} \).
Remark 1 To show that $x(\mu)$ is bounded away from 0, we can also let $x_0$ be the unique solution of (1) in the interval $0 < x < 1$ and prove that $x(\mu) \geq x_0$.

Let $F(\mu) = \lambda x_0^r - \mu - (\lambda x_0 - \mu)^r, 0 < \mu < 1/2$. It suffices to show that $F(\mu) < 0$ for $\mu > 0$. We have $F(0) = 0, F'(0) = (r - 1)x_0^r + rx_0^{r-1} - 1 = 0$ and $F''(\mu) = -r(r - 1)(\lambda x_0 - \mu)^{r-2}(x_0 + 1)^2 < 0$. It follows that $F(\mu) < 0$.

References