Open Mapping Theorem

**Theorem 1 (Open Mapping Theorem)** Let $X, Y$ be Banach spaces, and $T : X \to Y$ a continuous linear map from $X$ onto $Y$. The $T$ is an open map.

**Proof.** We shall denote $X_r = S(0, r)$, the open ball in $X$ centered at 0 with radius $r$, and $Y_r$ the same in $Y$. It suffices to prove that for any $X_r$, $T(X_r)$ contains a $Y_s$. Assuming that this has been proved, let $G$ be a non-void open set in $X$, and let $x + X_a \subset G$. Let $Y_b$ be such that $Y_b \subset T(X_a)$. Then

$$T(G) \supset T(x + X_a) = Tx + T(X_a) \supset Tx + Y_b,$$

which shows that $T(G)$ contains a neighborhood of every one of its points, and hence is open.

Since $X = \bigcup_{n=1}^{\infty} nX_r/2$, and $Y = T(X) = \bigcup_{n=1}^{\infty} nT(X_r/2)$, by Baire category theorem, one of the sets $nT(X_r/2)$ contains a non-void open set. Since the map $y \to ny$ is a homeomorphism in $Y$, $\overline{T(X_r/2)}$ contains a non-void open set $V$. Thus,

$$\overline{T(X_r)} \supset \overline{T(X_r/2)} - \overline{T(X_r/2)} \supset \overline{T(X_r/2)} - \overline{T(X_r/2)} \supset V - V.$$ 

Since a map of the form $y \to a - y$ is a homeomorphism, the set $a - V$ is open. Since the set $V - V = \bigcup_{a \in V} (a - V)$ is the union of open sets, it is open, contains 0, and hence contains a $Y_t$.

Let $\epsilon_0 = r/2$, and let $\epsilon_i > 0$ be such that $\sum_{i=1}^{\infty} \epsilon_i < \epsilon_0$. Then, according to the result stated in the preceding paragraph, there is a sequence $\{t_i, i = 0, 1, \ldots\}$ with $t_i > 0, t_i \to 0$, and such that

$$TX_{\epsilon_i} \supset Y_{t_i}, i = 0, 1, \ldots$$

(1)

Let $y \in Y_{t_0}$. It will be shown that there is an $x \in X_r$ such that $Tx = y$. From (1), with $i = 0$, it is seen that there is an $x_0 \in X_{\epsilon_0}$ such that $\|y - Tx_0\| < t_1$. Since $y - Tx_0 \in Y_{t_1}$, from (1), with $i = 1$, there is an $x_1 \in X_{\epsilon_1}$ with
\[\|y - Tx_0 - Tx_1\| < t_2.\] Continuing in this manner, a sequence \(\{x_n\}\) may be defined for which \(x_n \in X_{\epsilon_n}\), and

\[\|y - T\left(\sum_{i=0}^{n} x_i\right)\| < t_{n+1}, \quad n = 0, 1, \ldots\]  

Let \(z_m = x_0 + \ldots + x_m\), so that for \(m > n\), \(\|z_m - z_n\| = \|x_{n+1} + \ldots + x_m\| < \epsilon_{n+1} + \ldots + \epsilon_m\). This shows that \(\{z_n\}\) is a Cauchy sequence, and that the series \(x_0 + x_1 + \ldots\) converges to a point \(x\) with

\[\|x\| = \lim_{n \to \infty} \|z_n\| \leq \lim_{n \to \infty} (\epsilon_0 + \epsilon_1 + \ldots + \epsilon_n) < 2\epsilon_0 = r.\]

Since \(T\) is continuous, it is seen from (2) that \(y = Tx\). Thus it has been shown that an arbitrary ball \(X_r\) about the origin in \(X\) maps onto a set \(TX_r\) which contains a ball \(Y_s = Y_{t_0}\) about the origin in \(Y\).

**Theorem 2** A continuous linear one-to-one map of one Banach space onto all of another has a continuous linear inverse.

**Proof.**

Let \(X, Y\) be Banach spaces and \(T\) a continuous linear one-to-one map with \(TX = Y\). Since \((T^{-1})^{-1} = T\) maps open sets onto open sets by Open Mapping Theorem, the map \(T^{-1}\) is continuous. Let \(y_1, y_2 \in Y, x_1, x_2 \in X, Tx_1 = y_1, Tx_2 = y_2\), and \(\alpha\) a scalar. Then,

\[T(x_1 + x_2) = Tx_1 + Tx_2 = y_1 + y_2, T\alpha x_1 = \alpha Tx_1 = \alpha y_1,\]

so that

\[T^{-1}(y_1 + y_2) = x_1 + x_2 = T^{-1}y_1 + T^{-1}y_2,\]

and \(T^{-1}(\alpha y_1) = \alpha x_1 = \alpha T^{-1}y_1\). These equations show that \(T^{-1}\) is linear.

**Remark 1** The open mapping theorem (Theorem 1) can also be derived from Theorem 2 (which is Theorem 2 p.229 in KF’s book), as follows:

Let \(K = \{x : Tx = 0\}\). Then \(K\) is closed linear subspace of \(X\). Let \(\tilde{T} : X/K \to Y\) be the map induced by \(T\). \(\tilde{T}\) is one-to-one and continuous, so by Theorem 2, it is an open map. Let \(\pi : X \to X/K\) be the natural projection. \(\pi\) is an open map (see http://math.gmu.edu/~tlim/Problem4Page141Math675.pdf).

So \(T = \tilde{T} \circ \pi\) is open.

**Lemma 1** Let \(X, Y\) be normed linear spaces. If there is a linear homeomorphism between \(X\) and \(Y\), then either both spaces are complete or both are incomplete.
Proof.
If $T$ is such a homeomorphism from $X$ to $Y$, then there exist positive constants $C, D$ such that

$$D\|x\| \leq \|Tx\| \leq C\|x\|.$$ 

It follows that a sequence is Cauchy (convergent to $x$) in $X$ if and only if it images under $T$ is Cauchy (convergent to $Tx$) in $Y$.

Example 1 Let $X$ be $C([a,b])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and $Y$ be $C([a,b])$ with the norm $\|f\| = \|f\|_\infty$. Let $T$ be the identity map from $X$ to $Y$. Then $T$ is continuous, but $T^{-1}$ is not continuous since $Y$ is not complete.

Example 2 Let $Y$ be an infinite dimensional real Banach space and let $B = \{y_i : i \in \kappa\}$ be a Hamel basis for $Y$ such that $\|y_i\| = 1$ for all $i$. Let $X$ be the set of functions $f$ from $\kappa$ to $\mathbb{R}$ such that $f(i) = 0$ for all but finitely many $i$’s. Equip $X$ with the norm defined by $\|f\| = \sum |f(i)|$. Then $X$ is an incomplete normed linear space. Define $T : X \to Y$ by $T(f) = \sum f(i)y_i$. Then $T$ is one-to-one continuous linear. $T^{-1}$ is not continuous since $X$ is incomplete (and $Y$ is complete).

Definition 1 Let $T$ be a linear map whose domain $D(T)$ is a linear subspace of a Banach space $X$, and whose range lies in a Banach space $Y$. The graph of $T$ is the set of all points in the product space $X \times Y$ of the form $(x, Tx)$ with $x \in D(T)$. The operator $T$ is said to be closed if its graph is closed in the product space $X \times Y$. An equivalent statement is as follows: The operator $T$ is closed if $x_n \in D(T), x_n \to x, Tx_n \to y$ imply that $x \in D(T)$ and $Tx = y$.

Theorem 3 (Closed Graph Theorem) A closed linear map defined on all of a Banach space, and with values in a Banach space, is continuous.

Proof.
Note first that the product $X \times Y$ of two Banach spaces is a Banach space, where the norm of $(x, y)$ is defined as $\|x\| + \|y\|$. The graph $G$ of $T$ is a closed linear subspace in this product space; hence it is a Banach space. The map $P_X : (x, Tx) \to x$ of $G$ onto $X$ is one-to-one, linear, and continuous. Hence by Theorem 2, its inverse $P_X^{-1}$ is continuous. Thus $T = P_YP_X^{-1}$ is continuous.

Definition 2 A family $F$ of functions which map one vector space $X$ into another vector space $Y$ is called total if $x = 0$ is the only vector in $X$ for which $f(x) = 0$ for all $f$ in $F$. 

3
**Theorem 4** Let $X, Y,$ and $Z$ be Banach spaces and let $F$ be a total family of continuous linear maps on $X$ to $Y$. Let $T$ be a linear map from $Z$ to $X$ such that $fT$ is continuous for every $f$ in $F$. Then $T$ is continuous.

**Proof.**
We shall show that $T$ is closed, and apply Closed Graph Theorem. Let $\lim_{n \to \infty} z_n = z$, and let $\lim_{n \to \infty} Tz_n = x$. Then $\lim_{n \to \infty} f(Tz_n) = f(x)$ for each $f \in F$, since each $f \in F$ is continuous. On the other hand, $\lim_{n \to \infty} f(Tz_n) = f(Tz)$, since each of the function $fT$ is continuous. Therefore,

$$f(Tz) = f(x) \text{ for } f \in F,$$

and, since $F$ is total, $Tz = x$. This proves that $T$ is closed, and Closed Graph Theorem gives the desired conclusion.