On Moduli of k-Convexity

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Abstract

In this paper, we establish the continuity of some moduli of k-convexity.

Let \( X \) be a Banach space. We shall denote \( X^* \) the dual space of \( X \) and \( B_X \) the unit ball of \( X \). Several moduli of convexity for the norm of \( X \) have been defined; the last two definitions in the following are valid for spaces having dimension \( \geq k \):

\[
\delta_X(\epsilon) = \inf \{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \epsilon\} \quad [2]
\]

\[
\delta_X^{(k)}(\epsilon) = \inf \{1 - \frac{\|x_1 + \cdots + x_{k+1}\|}{k+1} : x_1, \cdots, x_{k+1} \in B_X, A(x_1, \cdots, x_{k+1}) \geq \epsilon\} \quad [10]
\]

\[
\Delta_X^{(k)}(\epsilon) = \inf \inf_{\|x\|=1} \sup_{\dim(Y)=k, \|y\|=1} \{\|x + \epsilon y\| - 1\} \quad [9],
\]

where

\[
A(x_1, \cdots, x_{k+1}) = \frac{1}{k!} \sup \left\{ \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \cdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_1, \cdots, f_k \in B_{X^*} \right\}.
\]

Evidently, by subtracting the first column from the other columns, the determinant can be replaced by

\[
\begin{vmatrix} f_1(x_2 - x_1) & \cdots & f_1(x_{k+1} - x_1) \\ \vdots & \cdots & \vdots \\ f_k(x_2 - x_1) & \cdots & f_k(x_{k+1} - x_1) \end{vmatrix}.
\]
Also \(A(x_1, \ldots, x_{k+1})\) can be thought of as the "volume" of the convex hull of \(x_1, \ldots, x_{k+1}\) since that is the case in Euclidean spaces.

\(X\) is called uniformly convex if \(\delta_X(\epsilon) > 0\) for \(\epsilon > 0\) and \(k\)-uniformly convex if \(\delta_X^{(k)}(\epsilon) > 0\) for \(\epsilon > 0\). Note that \(\delta_X(\epsilon) = \delta_X^{(1)}(\epsilon)\); so 1-uniform convexity coincides with uniform convexity. Lin [8] proved that \(\Delta_X^{(k)}(\epsilon) > 0\) for \(\epsilon > 0\) is equivalent to \(k\)-uniform convexity. Gurarii [5] proved that \(\delta_X(\epsilon)\) is continuous on \([0, 2]\) and there exist spaces of which \(\delta_X(\epsilon) = 0\) for \(0 \leq \epsilon < 2\) and \(\delta_X(2) = 1\). The continuity problem of \(\delta_X^{(k)}\) was mentioned in Kirk [6]. Let \(\mu_X^{(k)} = \sup\{A(x_1, \ldots, x_{k+1}) : x_1, \ldots, x_{k+1} \in B_X\}\). Note that \(\mu_X^{(1)} = 2\). In this note we prove that \(\delta_X^{(k)}(\epsilon)\) is continuous on \([0, \mu_X^{(k)}]\). It is quite evident that \(\Delta_X^{(k)}(\epsilon)\) satisfy the Lipschitz condition with constant 1.

**Definition 1** Let \(k \geq 1\) and \(0 \leq a < b \leq \infty\). A function \(f(\epsilon)\) on \((a, b)\) is called \(k\)-convex if

\[
f((\lambda \epsilon_2^{1/k} + (1 - \lambda) \epsilon_1^{1/k})^k) \leq \lambda f(\epsilon_2) + (1 - \lambda) f(\epsilon_1)
\]

for every \(\epsilon_1, \epsilon_2 \in (a, b), 0 \leq \lambda \leq 1\).

Obviously 1-convexity is simply the ordinary convexity.

**Lemma 2** Let \(0 \leq a < b \leq \infty\) and let \(f\) be a nondecreasing \(k\)-convex function on \((a, b)\) with \(M = \sup_{a < x < y < b} (f(y) - f(x)) < \infty\). Let \(\epsilon_1 < \epsilon_2, \epsilon_1, \epsilon_2 \in (a, b)\). Then

\[
\frac{f(c) - f(\epsilon_1)}{c - \epsilon_1} \leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k}) \epsilon_1^{1-1/k}}
\]

for every \(\epsilon_1 < c < \epsilon_2\).

**Proof.** Let \(z(x), \epsilon_1 \leq x \leq \epsilon_2\) be the function whose graph is defined by

\[
\begin{align*}
x &= (\lambda \epsilon_2^{1/k} + (1 - \lambda) \epsilon_1^{1/k})^k \\
y &= \lambda f(\epsilon_2) + (1 - \lambda) f(\epsilon_1)
\end{align*}
\]

By direct computation, we have

\[
z'(x) = \frac{f(\epsilon_2) - f(\epsilon_1)}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})(\lambda \epsilon_2^{1/k} + (1 - \lambda) \epsilon_1^{1/k})^{k-1}} \leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k}) \epsilon_1^{1-1/k}}.
\]

If \(\epsilon_1 < c < \epsilon_2\), then by the \(k\)-convexity of \(f\) and Mean-Value Theorem,

\[
\frac{f(c) - f(\epsilon_1)}{c - \epsilon_1} \leq \frac{z(c) - z(\epsilon_1)}{c - \epsilon_1} = z'(\psi) \leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k}) \epsilon_1^{1-1/k}}
\]

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The inequality in the following lemma is a consequence of a more general result proved in Bernal-Sullivan [1].

**Lemma 3** Let $X$ be a Banach space and $x_1, \ldots, x_{k+1} \in X$. Then

$$A(x_1, \ldots, x_{k+1}) \leq \frac{1}{k!} k^{k/2} \|x_2 - x_1\| \cdots \|x_{k+1} - x_1\|.$$  

**Proof.** Hadamard inequality says that if $r_1, r_2, \ldots, r_k$ are the rows (or columns) of a $k \times k$ matrix, then

$$\det(r_1, r_2, \ldots, r_k) \leq \|r_1\|_2 \|r_2\|_2 \cdots \|r_k\|_2.$$  

Here $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^k$. Since the Euclidean norm of the $j$-th column of the determinant in (1) is $\leq k^{1/2} \|x_j - x_1\|$, the inequality follows.

The inequality in the next theorem for the case $k = 1$ improves the one obtained in [5]. The general idea is similar to that in Goebel [3]. However, the reader should be aware that the assertion of Lemma 1 in that paper (that $\delta(\epsilon)$ is convex) is incorrect; a counterexample can be found in [7] or [4].

**Theorem 4** Let $X$ be a Banach space. Then

$$\frac{\delta_X^{(k)}(c) - \delta_X^{(k)}(\epsilon_1)}{c - \epsilon_1} \leq \frac{1}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k}) \epsilon_1^{1-1/k}}.$$  

for every $0 < \epsilon_1 < \epsilon < \epsilon_2 < \mu_X^{(k)}$.

**Proof.** For simplicity, in the following we will consider $k = 2$ and will indicate how to generalize to general $k$. Note that if $A(x_1, x_2, x_3) > 0$, then $x_2 - x_1$ and $x_3 - x_1$ are linearly independent.

For unit vectors $u, u_{21}, u_{31}$ and $u_{32}$ in $X$, with $\{u_{21}, u_{31}\}$ linearly independent, consider the set

$$N(u, u_{21}, u_{31}, u_{32}; \epsilon) = \{(x_1, x_2, x_3) \in X^3 : x_1 + x_2 + x_3 = \lambda u, x_2 - x_1 = \lambda_{21} u_{21}, x_3 - x_1 = \lambda_{31} u_{31}, x_3 - x_2 = \lambda_{32} u_{32} \text{ for some } \lambda, \lambda_{21}, \lambda_{31}, \lambda_{32} \geq 0 \text{ and } A(x_1, x_2, x_3) \geq \epsilon\}$$

and define

$$\delta(u, u_{21}, u_{31}, u_{32}; \epsilon) = \inf \{1 - \frac{\|x_1 + x_2 + x_3\|}{3} : (x_1, x_2, x_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon)\}.$$
Obviously, $\delta(u, u_{21}, u_{31}; u_{32}; \epsilon)$ is nondecreasing and has values in $[0, 1]$. If $(x_1, x_2, x_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon_1), (y_1, y_2, y_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon_2)$ and

$$x_1 + x_2 + x_3 = \lambda u, \quad x_2 - x_1 = \lambda_{21} u_{21}, \quad x_3 - x_1 = \lambda_{31} u_{31}, \quad x_3 - x_2 = \lambda_{32} u_{32}$$

$$y_1 + y_2 + y_3 = \alpha u, \quad y_2 - y_1 = \alpha_{21} u_{21}, \quad y_3 - y_1 = \alpha_{31} u_{31}, \quad y_3 - y_2 = \alpha_{32} u_{32}$$

for some $\lambda, \lambda_{ij}, \alpha, \alpha_{ij} \geq 0$, then by linear independence of $\{u_{21}, u_{31}\}$, there exists $c \geq 0$ such that

$$\alpha_{21} = c\lambda_{21}, \alpha_{31} = c\lambda_{31}, \alpha_{32} = c\lambda_{32}$$

Indeed, $\lambda_{32} u_{32} = x_3 - x_2 = (x_3 - x_1) - (x_2 - x_1) = \lambda_{31} u_{31} - \lambda_{21} u_{21}$ and $\alpha_{32} u_{32} = \alpha_{31} u_{31} - \alpha_{21} u_{21}$ imply

$$(\alpha_{32} \lambda_{31} - \lambda_{32} \alpha_{31}) u_{31} - (\alpha_{32} \lambda_{21} - \lambda_{32} \alpha_{21}) u_{21} = 0$$

from which it follows $\frac{\alpha_{31}}{\lambda_{31}} = \frac{\alpha_{32}}{\lambda_{32}} = \frac{\alpha_{21}}{\lambda_{21}}$.

Let

$$C(u_{21}, u_{31}) = \sup\{\begin{vmatrix} f_1(u_{21}) & f_1(u_{31}) \\ f_2(u_{21}) & f_2(u_{31}) \end{vmatrix} : f_1, f_2 \in B_X\}$$

Then $A(x_1, x_2, x_3) = \lambda_{21} \lambda_{31} C(u_{21}, u_{31}) \geq \epsilon_1$ and $A(y_1, y_2, y_3) = c^2 \lambda_{21} \lambda_{31} C(u_{21}, u_{31}) \geq \epsilon_2$.

For $0 \leq \zeta \leq 1$, let $z_i = \zeta x_i + (1 - \zeta)y_i, i = 1, 2, 3$. Then

$$z_2 - z_1 = (\zeta \lambda_{21} + (1 - \zeta)\epsilon_{21}) u_{21} = (\zeta + (1 - \zeta)c)\epsilon_{21} u_{21},$$

$$z_2 - z_1 = (\zeta + (1 - \zeta)c)\lambda_{31} u_{31},$$

$$z_3 - z_2 = (\zeta + (1 - \zeta)c)\lambda_{32} u_{32},$$

$$z_1 + z_2 + z_3 = (\epsilon \lambda + (1 - \zeta)\alpha) u,$$

$$A(z_1, z_2, z_3) = (\zeta + (1 - \zeta)c)^2 \lambda_{21} \lambda_{31} C(u_{21}, u_{31}) \geq (\zeta \epsilon_1^{1/2} + (1 - \zeta)\epsilon_2^{1/2})^2$$

and

$$1 - \frac{\|z_1 + z_2 + z_3\|}{3} = 1 - \frac{\|\zeta(x_1 + x_2 + x_3) + (1 - \zeta)(y_1 + y_2 + y_3)\|}{3}$$

$$= 1 - \frac{\|\zeta \lambda u + (1 - \zeta)\alpha u\|}{3}$$

$$= 1 - \frac{\zeta \lambda + (1 - \zeta)\alpha}{3}$$

$$= \zeta(1 - \frac{\lambda}{3}) + (1 - \zeta)(1 - \frac{\alpha}{3})$$

$$= \zeta(1 - \frac{\|x_1 + x_2 + x_3\|}{3}) + (1 - \zeta)(1 - \frac{\|y_1 + y_2 + y_3\|}{3})$$

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Hence
\[ \delta(u, u_{21}, u_{31}, u_{32}; (\zeta \epsilon_1^{1/2} + (1 - \zeta) \epsilon_2^{1/2})^2) \leq \zeta \delta(u, u_{21}, u_{31}, u_{32}; \epsilon_1) + (1 - \zeta) \delta(u, u_{21}, u_{31}, u_{32}; \epsilon_2) \]

Since
\[ \delta_X^{(2)}(\epsilon) = \inf \{ \delta(u, u_{21}, u_{31}, u_{32}; \epsilon) : \|u\| = \|u_{21}\| = \|u_{31}\| = \|u_{32}\| = 1, \{u_{21}, u_{31}\} \text{ linearly independent} \}, \]

and the inequality in Lemma 2 is preserved under passing to infimum, inequality (2) for \( k = 2 \) follows.

For general \( k \), we have \( \left(\frac{k+1}{2}\right) + 1 \) unit vectors \( u, u_{21}, \cdots \) and the proof is similar to the one above.

**Corollary 5** Let \( X \) be a Banach space. Then \( \delta_X^{(k)}(\epsilon) \) is continuous on \( [0, \mu_X^{(k)}) \).

**Proof.**
Take \( \|x_1\| = 1 \) and \( x_2, \cdots, x_{k+1} \) in a small ball centered at \( x_1 \). Then by Lemma 3 \( A(x_1, \cdots, x_{k+1}) \) is small. Since \( 1 - \frac{\|x_1 + \cdots + x_{k+1}\|}{k+1} \) is close to 0, we see that \( \delta_X^{(k)}(\epsilon) \) is continuous at 0.
Continuity of \( \delta_X^{(k)}(\epsilon) \) on \( (0, \mu_X^{(k)}) \) follows immediately from the inequality (2).

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**References**


