Math 302 HW 1 Solutions.

1. This mathematical system consists of 3 points P, Q, R,
   3 lines \{P,Q\}, \{P,R\}, \{Q,R\} (each containing two points) and
   one plane \{P,Q,R\} (containing 3 points only).

Axiom I. Any two distinct points O and P lie on a unique line OP. Each
line passes through at least two distinct points.

In our system, P,Q lie in \{P,Q\},
P,R lie in \{P,R\}
O,R lie in \{Q,R\}

and each line has exactly two distinct pts.

So Axiom I is satisfied.

Axiom II. Any three noncollinear points O, P, and Q lie in a unique plane
OPQ. Each plane passes through at least three noncollinear points.

In our system, the only three noncollinear pts are P, Q, R

and they lie in the plane \{P,Q,R\}. So Axiom II is satisfied.

Axiom III. If two distinct points O and P lie in a plane \alpha, then the
line OP lies entirely in \alpha.

In our system \{P,Q\}, \{P,R\}, \{Q,R\} \subset \{P,Q,R\}; so Axiom III

is satisfied.

Axiom IV. If two planes pass through a point O, then they pass through
another point P \neq O.

In our system, any two planes must be the same plane \{P,Q,R\}.

Since it contains 3 points, Axiom IV is satisfied.

Axiom V. There exist noncoplanar points O, P, Q, and R such that
O, P, and Q are noncollinear and O \neq P.

In our system, all points lie in the plane \{P,Q,R\}.

So there is no noncoplanar points.

So our system fails to satisfy Axiom V.
2. Under Axioms I1-I5, is it possible to prove that given any point \( P \), there are at least 4 distinct planes through \( P \)? Why?

Solution: It is enough to show a system which satisfies Axioms I1-I5 in which there is a point not lying in 4 distinct planes.

The system consists of 4 distinct points \( O, P, Q, R \)
6 lines \( \{O, P\}, \{O, Q\}, \{O, R\}, \{P, Q\}, \{P, R\}, \{Q, R\} \)
and 4 planes \( \{O, P, Q\}, \{O, P, R\}, \{O, Q, R\}, \{P, Q, R\} \)
(Each line contains only 2 points; each plane contains only 3 points)

One checks easily that this system satisfies Axioms I1-I5.

The point \( O \) lies in only 3 distinct planes.

3. Prove that for any real numbers \( x, y, z \), \( z \) is between \( x \) and \( y \) iff

\[ |x - y| = |x - z| + |z - y|. \]

Solution: By definition, "\( z \) is between \( x \) and \( y \)" means \( x \leq z \leq y \) or \( y \leq z \leq x \).

\( \Rightarrow \): Suppose \( x \leq z \leq y \). Then

\[ |x - y| = y - x \]
\[ |x - z| = z - x \]
\[ |z - y| = y - z \].

So \( |x - z| + |z - y| = z - x + y - z = -x + y = y - x = |x - y| \).

Similarly, the case for \( y \leq z \leq x \) is proved similarly.

\( \Leftarrow \): Suppose \( z \) is not between \( x \) and \( y \). Then either (a) \( z < x \leq y \)

or (b) \( x \leq y < z \) or (c) \( z < y \leq x \) or (d) \( y < x < z \).

In each case

In cases (a) and (d), \( |z - y| > |x - y| \)

\[ \Rightarrow |z - y| + |x - z| > |x - y| \]

In cases (b) and (c) \( |x - z| > |x - y| \)

\[ \Rightarrow |x - z| + |z - y| > |x - y| \]

So \( |x - z| + |z - y| \neq |x - y| \).
4. We need to show that $\equiv$ is reflexive, symmetric and transitive.

For any real number $x$, $x \equiv x$ is true $\because x - x = 0$ is rational.

Suppose $x \equiv y$. Then $x - y$ is rational. So $y - x = -(x - y)$ is rational and $y \equiv x$.

Suppose $x \equiv y$ and $y \equiv z$. Then $x - y$, $y - z$ are rational.

So $x - z = (x - y) + (y - z)$ is rational (sum of two rational nos is rational).

For each $x$, $[x]$  
Recall that for each real number $a$, $[a]$ is the set of all numbers equivalent to $a$; each of these is an equivalence class.
Two classes $[a]$, $[b]$ are identical iff $a \equiv b$. So for instance $[0] = \{x : x \equiv 0\} = \{x : x - 0 \text{ is rational}\} = \{x : x \text{ is rational}\}$ is identical with $[\frac{1}{2}] = \{x : x \equiv \frac{1}{2}\} = \{x : x - \frac{1}{2} \text{ is rational}\} = \{x : x \text{ is rational}\}$ (by $x - \frac{1}{2}$ is rational

$\iff x$ is rational

General $[a]$ is identical with $[b]$ iff $a - b$ is rational. So some of the distinct equivalence classes are $[0]$, $[\sqrt{2}]$, $[\sqrt{3}]$, $[\sqrt{5}]$, $[\pi]$, ...
5. Let $P$ be a given point. Axiom 15 implies that there is at least one point $Q$ distinct from $P$. So Axiom 31 says let $x = P$ (Axiom 11).

Let $c$ be a scale on $l$ such that $c(P) = 0$. Since $c$ is a distinct one-to-one map from $l$ to $\mathbb{R}$, there exist points $Q, R$ on $l$ such that $c(Q) = d$ and $c(R) = -d$. Then

\[ PQ = |c(P) - c(Q)| = |0 - d| = d \]
\[ PR = |c(P) - c(R)| = |0 - (-d)| = d. \]

There must be a point $W$ not on $l$ (otherwise Axiom 15 is violated).

Let $m$ be the line $\overrightarrow{PQW}$. As in the last paragraph, there exist distinct points $S, T$ on $m$ such that $PS = PT = d$.

Since $\emptyset \subseteq P \cup \overrightarrow{PQW} = \{P\}$ (Thm 3.2), $S, T, Q, R$ are all distinct.