Review: Polar Coordinates

P has rectangular coordinates 
(x, y).

Conversions:
(r, θ) → (x, y)

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

(x, y) → (r, θ)

\[ r^2 = x^2 + y^2 \] or \[ r = \sqrt{x^2 + y^2} \] if \( r \geq 0 \)
\[ \tan \theta = \frac{y}{x} \] or \[ \theta = \tan^{-1} \left( \frac{y}{x} \right) \] if \( x, y \) is in the 1st quadrant.

Example: Find the polar coordinates \((r, \theta)\) with \( r > 0 \) and \( 0 \leq \theta < 2\pi \) of the point \( P \) whose rectangular coordinates are \( x = -1, \ y = 2 \).

Solution:

\[ r = \sqrt{(-1)^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5} \]

From \( \triangle OPA \), \( \pi - \theta \) is an acute angle and \( \tan(\pi - \theta) = \frac{2}{1} = 2 \).

So \( \pi - \theta = \tan^{-1} 2 \)

\[ \therefore \theta = \pi - \tan^{-1} 2 \approx 2.0344 \text{ (rad.)} \approx 116.565^\circ \]
Rectangular coordinates

$P: (x, y, z)$

Cylindrical coordinates

(Polar coordinates in 3-d space)

$P: (r, \theta, z)$

\begin{align*}
y & = x^2 + y^2 \\
\tan \theta & = \frac{y}{x} \\
x & = r \cos \theta \\
y & = r \sin \theta \\
z & = z
\end{align*}

2x. Find the cylindrical coordinates of the point $P$ whose rectangular coordinates are $x = -1$, $y = -3$, $z = 4$; make $r > 0$ and $0 \leq \theta < 2\pi$.

So

\begin{align*}
y & = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-3)^2} = \sqrt{1 + 9} \\
& = \sqrt{10}
\end{align*}

From $\triangle OPA$

\[
\tan \left( \frac{3\pi}{2} - \theta \right) = \frac{1}{3}
\]

\[
\frac{3\pi}{2} - \theta = \tan^{-1} \left( \frac{1}{3} \right) \approx 0.3218
\]

\[
\theta = \frac{3\pi}{2} - 0.3218 \approx 4.39
\]

\[
\approx 251.57^\circ
\]

\[
z = 4 \quad (\sqrt{10}, 4.39, 4) \quad (\text{Ans}.)
\]
3x. Write the equation \( z^2 = x^2 + y^2 \) (a 45° cone) in cylindrical coordinates.

**Sol.** Since \( r^2 = x^2 + y^2 \), we have
\[
 z^2 = r^2 
\]  
(Ans.)

3x. Identify the surface \( r = 2 \) in cylindrical coordinates.

**Sol.**

\[
 x^2 + y^2 = 4 
\]

3x. Identify the surface \( r = 4 \sin \theta \) in cylindrical coordinates.

**Sol.** Multiplying both sides of \( r = 4 \sin \theta \) by \( r \):
\[
 r^2 = 4r \sin \theta 
\]

Now recall that \( r^2 = x^2 + y^2 \) and \( r \sin \theta = y \) and you get
\[
 x^2 + y^2 = 4y 
\]

\[
 x^2 + y^2 - 4y = 0 
\]

\[
 x^2 + y^2 - 4y + 4 = 4 
\]

\[
 x^2 + (y-2)^2 = 2^2 
\]
2x. Write the equation of the sphere $x^2 + y^2 + z^2 = 4$ in cylindrical coordinates.

Sol. $r^2 + z^2 = 4$ (Ans.)

Spherical Coordinates

$p$: distance from $P$ to the origin.

$\phi$: angle from the positive $z$-axis to the vector $\overrightarrow{OP}$, i.e., angle between $\mathbf{k}$ and $\overrightarrow{OP}$.

The same $\theta$ as in cylindrical coordinates.

Spherical coordinates of $P$: $(p, \theta, \phi)$
From $\triangle OPA$, we get
\[ z = \rho \cos \phi \]
\[ r = \rho \sin \phi \]
So we have these equations:
\[ x = r \cos \theta = \rho \sin \phi \cos \theta \]
\[ y = r \sin \theta = \rho \sin \phi \sin \theta \]
\[ z = \rho \cos \phi. \]

\[ \rho = \sqrt{x^2 + y^2 + z^2} \]
\[ \tan \theta = \frac{y}{x} \]
\[ \phi = \cos^{-1} \left( \frac{z}{\rho} \right) \]

Ex. If the rectangular coordinates of $P$ are $x=1, y=-3, z=-2$, find its spherical coordinates $(\rho, \theta, \phi)$, with $\rho > 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$.

So\[ \rho = \sqrt{1^2 + (-3)^2 + (-2)^2} = \sqrt{1 + 9 + 4} = \sqrt{14} \]
\[ \tan (2\pi - \theta) = \frac{3}{1} = 3 \]
\[ 2\pi - \theta = \tan^{-1} 3 \]
\[ \theta = 2\pi - \tan^{-1} 3 \approx 5.034 \quad (\approx 288.43^\circ) \]
\[ \phi = \cos^{-1} \left( \frac{-2}{\sqrt{14}} \right) \approx 2.1347 \quad (\approx 122.31^\circ) \]
8x. If \( P \) has spherical coordinates \((\rho, \theta, \phi) = (2\sqrt{2}, \frac{3\pi}{2}, \frac{\pi}{2})\), find its rectangular coordinates.

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta = 2\sqrt{2} \sin \frac{3\pi}{2} \cos \frac{\pi}{2} = 2\sqrt{2} \cdot 0 = 0 \\
y &= \rho \sin \theta \sin \phi = 2\sqrt{2} \sin \frac{3\pi}{2} \sin \frac{\pi}{2} = 2\sqrt{2} \cdot (-1) = -2\sqrt{2} \\
z &= \rho \cos \phi = 2\sqrt{2} \cos \frac{\pi}{2} = 0
\end{align*}
\]

\[\therefore (x, y, z) = (0, -2\sqrt{2}, 0) \quad (\text{Ans.})\]

9a. Write the equation \( x^2 + y^4 + z^4 = 4 \) (a sphere) in spherical coordinates.

\[\rho^2 = 4 \quad \therefore \rho = 2.\]

9b. Write the equation \( x^2 + y^2 = 2z \) in spherical coordinates.

\[
\begin{align*}
r^2 &= 2z \\
(\rho \sin \phi)^2 &= 2\rho \cos \phi \\
\rho^2 \sin^2 \phi &= 2\rho \cos \phi \\
\rho \sin^2 \phi &= 2 \cos \phi
\end{align*}
\]

\[\therefore \rho = 2 \cos \phi \cot \phi \quad (\because \cos \phi = \frac{1}{\sin \phi})\]
3x. Identify the surface $\rho = 2 \sin \theta \sin \phi$ in spherical coordinates.

Sol: Multiply both sides by $\rho$:

$$\rho^2 = 2 \rho \sin \theta \sin \phi$$

$$= 2 \rho \sin \phi \sin \theta$$

Now recall that $\rho^2 = x^2 + y^2 + z^2$, $r = \rho \sin \phi$ and $y \sin \theta = y$. Then

$$x^2 + y^2 + z^2 = 2y$$

$$x^2 + y^2 - 2y + z^2 = 0$$

$$x^2 + y^2 - 2y + 1 + z^2 = 1$$

$$x^2 + (y - 1)^2 + z^2 = 1$$

Sphere with center at $(0, 1, 0)$ with radius $= 1$.

3x. Identify the surface $\phi = \frac{\pi}{4}$ in spherical coordinates.

Sol.

The upper part of the $45^\circ$ cone

$$z^2 = x^2 + y^2.$$
Parametric equations of curves on the plane:

e.g. \[ \begin{align*}
    x &= 2 + 3t \\
    y &= 1 - 4t \\
    z &= 2 - 7t
\end{align*} \] a line

\[ \begin{align*}
    x &= 2 \cos t \\
    y &= 2 \sin t \\
    z &= t
\end{align*} \] a circle

radius = 2, center = (0, 0).

Parametric equations of curves in the x-y-z space:

e.g. \[ \begin{align*}
    x &= 2 + 3t \\
    y &= 1 - 4t \\
    z &= 2 - 7t
\end{align*} \] a line

\[ \begin{align*}
    x &= 2 \cos t \\
    y &= 2 \sin t \\
    z &= t
\end{align*} \] a spiral on the cylinder \( x^2 + y^2 = 4 \)

\[ t = 0: x = 2, y = 0, z = 0 \]

We can think of these curves as vector-valued functions. e.g. For the last curve

\[ r(t) = 2 \cos t \hat{i} + 2 \sin t \hat{j} + t \hat{k} \]

\[ = < 2 \cos t, 2 \sin t, t > \]

defines a vector-valued function.
If $t$ is the time and $(x, y, z)$ is the position of an object at time $t$, then

$r(t)$ is the position vector of the object at time $t$.

At time $t=0$

position vector $p(0) = <2, 0, 0>$

At time $t=\frac{\pi}{4}$

position vector is $p(\frac{\pi}{4}) = <\sqrt{2}, \sqrt{2}, \frac{\pi}{4}>$

More complicated curves like $x = \cos t$, $y = \sin t$, $z = \sin 5t$ are difficult to draw by hand; use a computer software e.g.

Maple:

Maple commands:

\[
\text{with(plots); spacecurve([cos(t), sin(t), sin(5*t)], t=0..2*Pi);} \]

Limit and continuity of vector-valued functions are defined componentwise:

if $r(t) = <f(t), g(t), h(t)>$

\[
\lim_{t \to a} r(t) = <\lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t)> \]

and $r(t)$ is continuous at $t=a$ if each $f(t), g(t), h(t)$ is continuous at $t=a$.

Derivative of $r(t)$ at $t=a$:

\[
r'(a) = \lim_{h \to 0} \frac{r(a+h) - r(a)}{h} \]
\[
\lim_{h \to 0} \left< \frac{f(a+h)-f(a)}{h} , \frac{g(a+h)-g(a)}{h} , \frac{h(a+h)-h(a)}{h} \right>
\]

\[
= \left< f'(a) , g'(a) , h'(a) \right>
\]

It is important to know that \( r'(a) \) is a vector tangent to the curve: \( x = f(t) \), \( y = g(t) \), \( z = h(t) \)

\[
\frac{r(a+h)-r(a)}{h}
\]

As \( h \to 0 \), \( \frac{r(a+h)-r(a)}{h} \) approaches to a vector tangent to the curve.

\( r'(a) \) is called the tangent vector of \( r(t) \) at \( t = a \).

\( r(t) = \left< 2 \cos t , 2 \sin t \right> \), \( r'(t) = \left< -2 \sin t , 2 \cos t \right> \)

\[
r'(\frac{\pi}{4}) = \left< -2 \sin \frac{\pi}{4} , 2 \cos \frac{\pi}{4} \right> \bigg|_{t = \frac{\pi}{4}}
\]

\[
= \left< -\sqrt{2} , \sqrt{2} \right>
\]
If \( t \) is the time and \( \mathbf{r}(t) \) the position vector of an object at time \( t \), then \( \mathbf{r}'(t) \) is the velocity vector of the object. ( \( \mathbf{r}'(a) \) : \( \mathbf{r}'(t) \) evaluated at \( t=a \), is the rect velocity of the object at time \( t=a \).) We will say more about this later.

Integral of \( \mathbf{r}(t) \):

\[
\int \mathbf{r}(t) \, dt = \langle \int x(t) \, dt, \int y(t) \, dt, \int z(t) \, dt \rangle
\]

E.g. if \( \mathbf{r}(t) = \langle t, t^2, t^3 \rangle \), then

\[
\int \mathbf{r}(t) \, dt = \langle \frac{t^2}{2} + c_1, \frac{t^3}{3} + c_2, \frac{t^4}{4} + c_3 \rangle
\]

\[= \langle \frac{t^2}{2}, \frac{t^3}{3}, \frac{t^4}{4} \rangle + \mathbf{c} \]

where \( \mathbf{c} = \langle c_1, c_2, c_3 \rangle \) is an arbitrary vector.

Basic Rules:

If \( \mathbf{u}(t), \mathbf{v}(t) \) are vector-valued functions and \( l(t) \) is a scalar function, then

\[
\frac{d}{dt} (\mathbf{u}(t) \pm \mathbf{v}(t)) = \frac{d}{dt} \mathbf{u}(t) \pm \frac{d}{dt} \mathbf{v}(t)
\]

\[
\frac{d}{dt} c \mathbf{u}(t) = c \frac{d}{dt} \mathbf{u}(t) \quad \text{for any scalar } c
\]
\[
\frac{d}{dt} \lambda(t) u(t) = \lambda(t) \frac{d}{dt} u(t) + \lambda'(t) u(t)
\]
\[
\frac{d}{dt} u(t) \cdot v(t) = u(t) \cdot \frac{d}{dt} v(t) + v(t) \cdot \frac{d}{dt} u(t)
\]
\[
\frac{d}{dt} u(t) \times v(t) = u(t) \times \frac{d}{dt} v(t) + \frac{d}{dt} u(t) \times v(t)
\]
\[
\frac{d}{dt} u'(t) = u'(t) \lambda(t) \lambda'(t)
\]

\[u(t) = \langle 1, t, t^2 \rangle \quad v(t) = \langle -t, 1, -2t \rangle\]

Then \[u'(t) = \langle 0, 1, 2t \rangle \quad v'(t) = \langle -1, 0, -2 \rangle\]

So \[\frac{d}{dt} u(t) \cdot v(t) = u(t) \cdot v'(t) + u'(t) \cdot v(t)\]
\[= \langle 1, t, t^2 \rangle \cdot \langle -1, 0, -2 \rangle + \langle 0, 1, 2t \rangle \cdot \langle -t, 1, -2 \rangle\]
\[= -1 - 2t^2 + 1 - 4t^2\]
\[= -6t^2\]

We can check this result by finding \[\frac{d}{dt} u(t) \cdot v(t)\] directly:

\[u(t) \cdot v(t) = \langle 1, t, t^2 \rangle \cdot \langle -t, 1, -2t \rangle\]
\[= 1 \cdot (-t) + t \cdot 1 + t^2 \cdot (-2t)\]
\[= -t + t - 2t^3 = -2t^3\]

\[\therefore \frac{d}{dt} u(t) \cdot v(t) = -2 \cdot 3t^2 = -6t^2\]

Same as above.
Ex. Prove that if \( |r(t)| = c \), a constant, then
\[ r'(t) \cdot r(t) = 0. \]

So \( |r(t)| = c \implies |r(t)|^2 = c^2 \)
\[ r(t) \cdot r(t) = c^2 \]

\[ \therefore \frac{d}{dt} r(t) \cdot r(t) = 0 \quad \text{by Rule} \]
\[ r'(t) \cdot r(t) + r(t) \cdot r'(t) \]
\[ = r'(t) \cdot r(t) + r'(t) \cdot r(t) \]
\[ \therefore 2r'(t) \cdot r(t) = 0 \]
\[ \therefore r'(t) \cdot r(t) = 0 \]

Interpretation: The result says if an object is travelling on a sphere in a circle centered at the origin, then the position vector is always perpendicular to the velocity vector, which seems to be obvious.