What is \( \iint_R 3 \, dA \), where \( R = [-1,1] \times [2,6] \)?

**Ans.** \( 3 \times 2 \times 4 = 24 \)

Why? b/c \( \iint_R 3 \, dA \) is the volume of the solid under \( z = 3 \) above \( [-1,1] \times [2,6] \), which is like a box with width = 2, length = 4 and height = 3.

If we have a more complicated function \( f(x,y) \), how do we evaluate \( \iint_R f(x,y) \, dA \)? A Riemann sum only gives an approximation.

Taking limit of Riemann sums is difficult. Fortunately, we can have the following

**Fubini's Theorem.** \( \iint_R f(x,y) \, dA = \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy \)

where \( R \) is the rectangle \([a,b] \times [c,d]\).

If \( f(x,y) \geq 0 \), each for each \( y \) fixed, \( \int_c^d f(x,y) \, dy \) is a cross section area; integrating this cross-section area from \( x = a \) to \( x = b \) gives the total volume. [Math 114 on Math 13, last chapter]. And this is roughly the proof of Fubini's Theorem. The second part is similar.
Ex. Evaluate \( \iint_R x + \sqrt{y} \, dA \) where \( R = [0, 1] \times [0, 2] \).

\[
\begin{align*}
\int_0^1 \int_0^2 x + \sqrt{y} \, dy \, dx &= \int_0^1 \left[ x y + \frac{y^{3/2}}{3/2} \right]_y^2 \, dx \\
&= \int_0^1 \left[ 2x + \frac{2 \cdot 2^{3/2}}{3/2} \right] \, dx \\
&= \left[ 2 \cdot \frac{x^2}{2} + \frac{2}{3} \cdot \frac{2^{3/2}}{x} \right]_0^1 \\
&= 16 + \frac{4 \cdot 2^{3/2}}{3} - 1 - \frac{2^{3/2}}{3} \\
&= 16 + \frac{16 \sqrt{2}}{3} - 1 - \frac{2 \sqrt{2}}{3} \\
&= 15 + \frac{12 \sqrt{2}}{3} = 15 + 4 \sqrt{2}.
\end{align*}
\]

OR. \( \int_0^2 \int_1^4 x + \sqrt{y} \, dx \, dy \)

\[
\begin{align*}
\int_0^2 \left[ \frac{x^2}{2} + \sqrt{y} x \right]_x=1 \, dy &= \int_0^2 \left[ \frac{x^2}{2} + 4 \sqrt{y} - \frac{x}{2} - \sqrt{y} \right] \, dy \\
&= \int_0^2 \frac{15}{2} + 3 \sqrt{y} \, dy \\
&= \left[ \frac{15}{2} y + 3 \cdot \frac{y^{3/2}}{3/2} \right]_0^2 \\
&= 15 + 2 \cdot 2^{3/2} - 0 \\
&= 15 + 2 \cdot 2^{3/2} = 15 + 4 \sqrt{2}.
\end{align*}
\]

Note that
Sometimes one way is easier than the other way.

Example: \( \int \int_R x e^{xy} \, dA \) where \( R = [0,1] \times [0,1] \).

\[
\int_0^1 \int_0^1 x e^{xy} \, dx \, dy
\]
is harder b/c \( \int_0^1 x e^{xy} \, dx \) requires integration by parts (Math II).

So let us try

\[
\int_0^1 \int_0^e x e^{xy} \, dy \, dx
\]

\[
= \int_0^e \left[ \frac{e^{xy}}{x} \right]_0^1 \, dy = x \left( \frac{e^x}{x} - \frac{e^0}{x} \right) = e^x - e^0
\]

\[
= e^x - 1
\]

b/c \( \int e^y \, dy = \frac{e^y}{a} \) if \( a \) is a constant.

\[
= e^x - 1
\]

(Math III).

\[
\int_0^1 \int_0^e x e^{xy} \, dy \, dx = \int_0^e e^x - 1 \, dx = [e^x - x]_0^1 = e - 1 - (e^0 - 0)
\]

\[
= e - 1 - (1 - 0)
\]

\[
= e - 1 - 1 = e - 2 \quad \text{(Ans.)}
\]

Try to evaluate \( \int \int_0^1 x e^{xy} \, dx \, dy \) using integration by parts.

If the function can be written as \( h(x)g(y) \) i.e. a product

of a fn. of \( x \) and a fn. of \( y \), then there is an easy formula.
\[ \iiint f(x,y) \, dx \, dy = \int_a^b \int_c^d f(x,y) \, dx \, dy \].

[Of course, you don't have to use this formula; only that using it makes your work shorter.]

\textbf{Proof:} \quad \iiint f(x,y) \, dx \, dy = \int_a^b \int_c^d f(x,y) \, dx \, dy

= \int_c^d g(y) \int_a^b f(x) \, dx \, dy

\text{where } g(y) \text{ can be factored out since it is a constant.}

Now in \[ \int_c^d g(y) \int_a^b f(x) \, dx \, dy \]  

\int_a^b f(x) \, dx \text{ is a constant, so it can be factored out to get}

\[ \int_a^b f(x) \, dx \cdot \int_c^d g(y) \, dy \]

\textbf{Example:} \quad \iiint x \, y^2 \, dx \, dy \quad \text{over } \quad R = [0,1] \times [2,3].

= \int_0^1 x^2 \, dx \cdot \int_2^3 y^2 \, dy

= \left[ \frac{x^3}{3} \right]_0^1 \cdot \left[ \frac{y^3}{3} \right]_2^3

= \frac{1}{3} \cdot \left( 9 - \frac{8}{3} \right) = \frac{1}{3} \cdot \frac{19}{3} = \frac{19}{9}.

Note: some sets cannot be written in the form \( h(x) \times g(y) \).

\text{E.g. } e^{x+y} \text{ NO, } e^{x+y} \text{ YES: } e^{x+y} = e^x \cdot e^y.
3X. Find the volume of the solid under \( \frac{x^2}{4} + \frac{y^2}{9} + z = 1 \) and above \([-1, 1] \times [-2, 2]\).

\[
\text{Volume} = \iint \left( 1 - \frac{x^2}{4} - \frac{y^2}{9} \right) \, dA
= \int_{-1}^{1} \int_{-2}^{2} \left( 1 - \frac{x^2}{4} - \frac{y^2}{9} \right) \, dy \, dx
\]

Note that \( 1 - \frac{x^2}{4} - \frac{y^2}{9} \) is even in \( y \) (replacing \( y \) by \( -y \) yields the same thing), so

\[
\int_{-2}^{2} \left( 1 - \frac{x^2}{4} - \frac{y^2}{9} \right) \, dy = 2 \int_{0}^{2} \left( 1 - \frac{x^2}{4} - \frac{y^2}{9} \right) \, dy
= 2 \left[ \left( 1 - \frac{x^2}{4} \right) \cdot \frac{y^3}{3} \right]_{y=0}^{y=2}
= 2 \left[ \left( 1 - \frac{x^2}{4} \right) \cdot 2 - \frac{8}{27} \right]
= 2 \left( 2 - \frac{x^2}{2} - \frac{8}{27} \right)
= 2 \left( \frac{4}{27} - \frac{8}{27} \right)
= \frac{2}{27} - \frac{x^2}{2}
\]

\[
\therefore \text{volume} = \int_{-1}^{1} \frac{2}{27} - \frac{x^2}{2} \, dx = 2 \int_{0}^{1} \frac{2}{27} - \frac{x^2}{2} \, dx
\]

even func. \[= 2 \left[ \frac{2}{27} x - \frac{x^3}{2} \right]_{0}^{1}
= 2 \left( \frac{2}{27} - \frac{1}{2} \right)
= 2 \left( \frac{4}{27} - \frac{9}{27} \right)
= 2 \left( \frac{9}{27} - \frac{9}{27} \right) = 2 \left( \frac{27}{27} \right)
\]

\[= \frac{166}{27}. \text{(Ans.)} \]