Problem: How to find the slope of the tangent line?

Solution: Find the slope of the secant line through $(a,f(a))$ and $(x,f(x))$ and let $x$ approach $a$. 

Slope of a secant line: \[ \text{slope} = \frac{f(b) - f(a)}{b - a} \] 

Tangent line at $(a,f(a))$ 

Secant line
Example: Find the slope of the tangent line to the curve \( y = x^2 \) at \( x = 1 \).

The secant line through \((1,1)\) and \((x, x^2)\) has

\[
\text{Slope} = \frac{x^2 - 1}{x - 1}, \quad \text{which simplifies to} \quad \frac{(x+1)(x-1)}{x-1} = x + 1.
\]

Now let \( x \) approach 1. Then \( x + 1 \) approaches 2.

So 2 must be the slope of the tangent line at \((1,1)\).

The fact that \( \frac{x^2 - 1}{x - 1} \) approaches 2 as \( x \) approaches 1 is written mathematically as

\[
\lim_{{x \to 1}} \frac{x^2 - 1}{x - 1} = 2
\]

Velocity as a limit.

If a car travels at a constant speed, say 50 m.p.h. (miles per hour), then its distance travelled after \( t \) hours
50 miles.

\[ s(t) \quad s = 50t \]

\[ \text{tangent line at every point of } s = 50t \text{ is the line } s = 50t \text{ itself. The slope is constant } ( = 50 ) \]

If the distance function \( s(t) \) is not a straight line, i.e., if the speed is not constant, then the instantaneous velocity at any time \( t = a \) is

\[ \lim_{t \to a} \frac{s(t) - s(a)}{t - a} \quad (1) \]

For example, if \( s(t) \) is the distance (in meters) traveled by a free-falling object after \( t \) seconds, then (by Physics)

\[ s(t) = 4.9 t^2 \]

So the instantaneous velocity of the object at time \( t = 2 \) sec.

is

\[ \lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2} \frac{4.9t^2 - 4.9 \times 4}{t - 2} \]
We can guess the limit from the following calculations:

\[
\begin{array}{c|c}
  t & \frac{4.9t^2 - 4.9t \cdot 4}{t - 4} \\
  \hline
  1 & 14.7 \\
  1.5 & 17.15 \\
  1.8 & 18.62 \\
  1.9 & 19.11 \\
  1.95 & 19.355 \\
  1.99 & 19.55 \\
\end{array}
\]

It appears that the limit is 19.6, and indeed it is as the following algebraic calculation shows:

\[
\frac{4.9t^2 - 4.9t \cdot 4}{t - 2} = \frac{4.9(t^2 - 4)}{t - 2} = \frac{4.9(t - 2)(t + 2)}{t - 2} = 4.9(t + 2)
\]

\[
\rightarrow 4.9(2 + 2) = 4.9 \times 4 = 19.6
\]

as \( t \to 2 \).

So the (instantaneous) velocity at time \( t = 2 \) sec. is 19.6 m/sec.

So we see that slope of tangent line, velocity are examples of "limit".
What is "limit"?

When we write \( \lim_{x \to 1} x + 1 = 2 \), we mean \( x + 1 \) approaches 2 as \( x \) approaches 1 — this seems to be obviously true.

But some limits are not so obvious. For instance, that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \) and \( \lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \) are not obvious at all!

\[
\lim_{x \to a} f(x) = L \quad \text{means} \quad \text{as } x \text{ approaches } a \quad \text{(but not equal to } a)\,, \ f(x) \text{ approaches (or equal to) } L\,.
\]

Notice that \( L \) need not be \( = f(a) \).

For instance in \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \), the function \( f(x) = \frac{x^2 - 1}{x - 1} \) is **undefined** at 1, i.e., \( f(1) \) is undefined! But the limit \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \) exists \( = 2 \).

So don’t always think that

\[
\lim_{x \to a} f(x) = f(a)
\]
[although this is true for many functions].

The following limits seem to be obvious:

\[
\lim_{x \to 1} x + 1 = 2 \\
\lim_{x \to 1} 3 = 3 \\
\lim_{x \to 2} x^2 - x + 2 = 4
\]

Let us check the last one:

<table>
<thead>
<tr>
<th>x</th>
<th>x^2 - x + 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2</td>
</tr>
<tr>
<td>1.5</td>
<td>2.75</td>
</tr>
<tr>
<td>1.8</td>
<td>3.44</td>
</tr>
<tr>
<td>1.9</td>
<td>3.71</td>
</tr>
<tr>
<td>1.95</td>
<td>3.8525</td>
</tr>
<tr>
<td>1.99</td>
<td>3.9701</td>
</tr>
<tr>
<td>1.995</td>
<td>3.985025</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|c|c|c|c|c}
\hline
x & \frac{x^2 - x + 2}{2} & 2.5 & 3.75 & 2.2 & 4.64 \\
\hline
2.1 & 4.31 & 2.05 & 4.025 & 2.01 & 4.0301 \\
1.995 & 3.985025 & 2.005 & 4.015025 & 2.001 & 4.003001 \\
\hline 2 & 4 & 2 & 4 & 2 & 4
\end{array}
\]

Note: You may write \( x^2 - x + 2 \to 4 \) as \( x \to 2 \)

But you don't write \( \lim_{x \to 2} x^2 - x + 2 \to 4 \). You write:

\[
\lim_{x \to 2} x^2 - x + 2 = 4
\]
2. Find the limit \( \lim_{x \to 1} f(x) \) if

\[
f(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1. \end{cases}
\]

\[\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}\]

When \( x \to 1 \) \( x \) is NOT allowed to be 1

\[\text{So } f(x) = \frac{x-1}{x^2-1}\]

No you can evaluate \( \lim_{x \to 1} \frac{x-1}{x^2-1} \) by algebraic means:

\[
\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1} \quad (\text{if } x \neq 1)
\]

\[\therefore \lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}\]

\[\text{Seems obvious}\]

\[\therefore \lim_{x \to 1} f(x) = \frac{1}{2}.\]

Note: This is an example that \( \lim_{x \to 1} f(x) \neq f(1) \) because \( f(1) = 2 \) by definition of \( f \).
Warning: Calculators can be misleading sometimes: see Example 2, Example 4, Example 5 in book.

[In example my calculator evaluates \( \frac{\sqrt{t^2+9}-3}{t^2} \) at \( t=0.0001 \) as 0, but the actual value is close to \( \frac{1}{6} \); this is due to round off errors in calculators.]

Example: Find \( \lim_{x \to 0} f(x) \) if

\[
f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}
\]

So ()

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1</td>
<td>-0.1</td>
<td>0</td>
</tr>
<tr>
<td>0.01</td>
<td>1</td>
<td>0.01</td>
<td>0</td>
</tr>
<tr>
<td>0.001</td>
<td>1</td>
<td>-0.001</td>
<td>0</td>
</tr>
<tr>
<td>0.0001</td>
<td>1</td>
<td>0.0001</td>
<td>0</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( \downarrow )</td>
<td>( \downarrow )</td>
<td>( \downarrow )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\( \lim_{x \to 0} f(x) \) does not exist \( \underline{b/c} \) \( f(x) \) approaches different values as \( x \) approaches 0 from the right or
One-sided limits

\[ \lim_{x \to a^+} f(x) \]

\text{means } x \text{ approaches } a \text{ from the right, } (x \to a^+)

\[ \lim_{x \to a^-} f(x) \]

\text{means } x \text{ approaches } a \text{ from the left, } (x \to a^-)

So in the last example where

\[ f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \]

we have \( \lim_{x \to 0^+} f(x) = 1 \) and \( \lim_{x \to 0^-} f(x) = 0 \)

Note that we have the following obvious fact:

\[ \lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = L \]
Ex. Let \( f \) be the function whose graph is as follows:

\[
\begin{array}{c|c}
2 & \\
1.7 & \\
1.4 & \\
\end{array}
\]

(Note that \( f(2) \) is undefined and \( f(4) = 1.4 \))

We have

\[
\lim_{x \to 2^-} f(x) = 2, \quad \lim_{x \to 2^+} f(x) = 1.4
\]

(\( \text{so}\) \( \lim_{x \to 2} f(x) \) \( \text{does not exist} \))

\[
\lim_{x \to 4^-} f(x) = 1.7, \quad \lim_{x \to 4^+} f(x) = 1.7
\]

\( \text{so}\) \( \lim_{x \to 4} f(x) = 1.7 \)

\( \text{(But} \ f(4) \neq 1.7, \ f(4) = 1.4! \)\)

Infinite limits

Look at \( \lim_{x \to 0^+} \frac{1}{x} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{1}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>10</td>
</tr>
<tr>
<td>0.01</td>
<td>100</td>
</tr>
<tr>
<td>0.001</td>
<td>1000</td>
</tr>
<tr>
<td>0.0001</td>
<td>10,000</td>
</tr>
<tr>
<td>0+</td>
<td>arbitrary large</td>
</tr>
</tbody>
</table>

As \( x \) approaches 0 from the right, \( \frac{1}{x} \) becomes arbitrary large.
We write \( \lim_{x \to 0^+} \frac{1}{x} = \infty \)

[NOTE: \( \infty \) is NOT a number; it is a concept.]

\[
\begin{array}{c|c}
 x & \frac{1}{x} \\
-0.1 & -10 \\
-0.01 & -100 \\
-0.001 & -1000 \\
-0.0001 & -10000 \\
\vdots & \vdots \\
0^- & (arbitrary large) \\
\end{array}
\]

We write \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \)

But for \( \frac{1}{x} \), \( \lim_{x \to 0^+} \frac{1}{x^2} = \lim_{x \to 0^-} \frac{1}{x^2} = \infty \); in this case we write \( \lim_{x \to 0} \frac{1}{x^2} = \infty \).

Other examples:

\[
\lim_{x \to 3^+} \frac{2}{x-3} = \infty, \quad \lim_{x \to 3^-} \frac{2}{x-3} = -\infty
\]

\[
\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty
\]

\[
\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty
\]

\[
\lim_{x \to 0^+} \ln x = -\infty
\]
\[
\text{If } \lim_{x \to a} f(x) \text{ or } \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) \\
\text{is infinity } (+\infty) \text{ or } -\text{infinity } (-\infty) \text{ we say} \\
\text{the function has a vertical asymptote at } x = a.
\]

\[
\text{Basic limit laws (theorems)}
\]

\[
\lim_{x \to a} c = c
\]

\[
\lim_{x \to a} x = a
\]

\[
\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

\[
\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x).
\]

\[
\lim_{x \to a} c f(x) = c \lim_{x \to a} f(x).
\]

\[
\lim_{x \to a} f(x) g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)
\]

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{provided } \lim_{x \to a} g(x) \neq 0.
\]

\[
\lim_{x \to a} f(x)^n = \left(\lim_{x \to a} f(x)\right)^n
\]

\[
\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad \text{where } n \text{ is a positive integer.}
\]
\[ \lim_{x \to a} x^n = a^n \]
\[ \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \]

Evaluate \[ \lim_{x \to 2} \frac{f(x)}{g(x)} \]

If

(i) \[ \lim_{x \to 2} f(x) = 0 \quad \text{and} \quad \lim_{x \to 2} g(x) = 2 \]

(ii) \[ \lim_{x \to 2} f(x) = 0 \quad \text{and} \quad \lim_{x \to 2} g(x) = 0 \]

(iii) \[ \lim_{x \to 2} f(x) = 1 \quad \text{and} \quad \lim_{x \to 2} g(x) = 0 \]

So,

(i) \[ \lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{0}{2} = 0 \]

(ii) Need more information to evaluate \[ \lim_{x \to 2} \frac{f(x)}{g(x)} \].

(iii) Need more information.

\[ \lim_{x \to 2} \frac{f(x)}{g(x)} \] is either \( \infty \) or \( -\infty \)

or does not exist.

Depending on the behavior of \( g(x) \) as \( x \to 2 \).

\[ f(x) = x - 1 \quad g(x) = x - 2 \]

then \[ \lim_{x \to 2} f(x) = 1, \quad \lim_{x \to 2} g(x) = 0 \]

but \[ \lim_{x \to 2} \frac{f(x)}{g(x)} \] does not exist

\[ (6/6 \quad \lim_{x \to 2^+} \frac{f(x)}{g(x)} = \infty \quad \lim_{x \to 2^-} \frac{f(x)}{g(x)} = -\infty) \]
While if \( f(x) = x-1 \), \( g(x) = (x-2)^2 \)

then
\[
\lim_{x \to 2} \frac{f(x)}{g(x)} = \infty.
\]

3x. \[
\lim_{x \to 5} 2x^2 + 3x + 4 = \lim_{x \to 5} 2x^2 - \lim_{x \to 5} 3x + \lim_{x \to 5} 4
\]
\[
= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + 4
\]
\[
= 2 \cdot 5^2 - 3 \cdot 5 + 4
\]
\[
= 50 - 15 + 4 = 39
\]

Generally,

if \( f(x) \) is any polynomial, then
\[
\lim_{x \to a} f(x) = f(a)
\]

3\( \infty \). \[
\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} x^3 + 2x^2 - 1}{\lim_{x \to -2} 5 - 3x} = \frac{(2)^3 + 2(-2)^2 - 1}{11}
\]
\[
= \frac{-8 + 8 - 1}{11} = -\frac{1}{11}
\]

Generally,

if \( f(x) \) is a rational function \( \frac{p(x)}{q(x)} \) and \( q(a) \neq 0 \), then
\[
\lim_{x \to a} f(x) = \frac{p(a)}{q(a)}.
\]
2x. Find \( \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} \) such that \( \frac{0}{0} \) limit laws do not apply.

\[ \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} \]

Expand the numerator:

\[ = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \]

Simplify the numerator:

\[ = \lim_{h \to 0} \frac{6h + h^2}{h} \]

Factor the numerator:

\[ = \lim_{h \to 0} \frac{h(6 + h)}{h} \]

Simplify:

\[ = \lim_{h \to 0} 6 + h \]

\[ = 6 \quad \text{(Ans.)} \]

3x. Find \( \lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} \) such that \( \frac{0}{0} \) limit laws do not apply.

\[ \lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} \]

\[ = \lim_{x \to 0} \frac{(\sqrt{x^2 + 9} - 3)(\sqrt{x^2 + 9} + 3)}{x^2 (\sqrt{x^2 + 9} + 3)} \]

Multiply both numerator and denominator by \( \sqrt{x^2 + 9} + 3 \):

\[ = \lim_{x \to 0} \frac{x^2 + 9 - 9}{x^2 (\sqrt{x^2 + 9} + 3)} \]

\[ = \lim_{x \to 0} \frac{x^2}{x^2 (\sqrt{x^2 + 9} + 3)} \]

\[ = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 9} + 3} \]

Limit laws apply:

\[ = \frac{1}{\sqrt{9} + 3} = \frac{1}{3 + 3} = \frac{1}{6} \]
Ex. Find \( \lim_{x \to 1} g(x) \), if

\[
g(x) = \begin{cases} 
x + 1 & x \neq 1 \\
\pi & x = 1
\end{cases}
\]

So, note that \( x \to 1 \Rightarrow x \neq 1 \) (\( x \) is not allowed to be 1)

\[
\lim_{x \to 1} g(x) = \lim_{x \to 1} x + 1 = 2 \quad \text{(Ans.)}
\]

\[g(1) = \pi \quad \text{(by the definition)} \quad \text{(Ans.)}
\]

Ex. Find \( \lim_{x \to 4} f(x) \), if

\[
f(x) = \begin{cases} 
\sqrt{x-4} & x > 4 \\
8 - 2x & x < 4
\end{cases}
\]

So, note that \( f(4) \) is undefined. But that does not mean that \( \lim_{x \to 4} f(x) \) does not exist.

In fact, \( \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = 0 \)

\[
\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} 8 - 2x = 0
\]

So, \( \lim_{x \to 4} f(x) = 0 \)
Limit law involving inequality.

If \( f(x) \leq g(x) \) then
\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).
\]

(The Squeeze theorem)

If \( f(x) \leq g(x) \leq h(x) \)
and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \)

then \( \lim_{x \to a} g(x) = L \).

Application: Show that \( \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 \)

(even though \( \lim_{x \to 0} \sin \frac{1}{x} \) does not exist).

\( \text{Sol.}\)

Since
\[-1 \leq \sin \frac{1}{x} \leq 1\]

we have \( -x^2 \leq x^2 \sin \frac{1}{x} \leq x^2 \) (multiply \( x^2 \) everywhere)

Now \( \lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0 \)

So by Squeeze theorem
\( \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0. \)
A function is said to be continuous at a number $a$ if
\[ \lim_{x \to a} f(x) = f(a) \text{, i.e.} \lim_{x \to a} f(x) = f(\lim_{x \to a} x) \]

**Examples:**

- **Not continuous at $a$:**
  - $b/c \lim_{x \to a} f(x)$ does not exist

- **Not continuous at $a$:**
  - $b/c \lim_{x \to a} f(x)$ exists but $f(a)$ is undefined.
  - $b/c \lim_{x \to a} f(x)$ exists but $\neq f(a)$.

**Facts:** Every polynomial function is continuous everywhere.

\[ (b/c \lim_{x \to a} f(x) = f(a) \text{ for polynomial}) \]

Every rational function is continuous everywhere where it is defined (i.e., at those numbers where the denominator is not 0).

\[ (b/c \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0) \]
3p. \(2x^3-x^2+x-1\) is cont. everywhere
\[
\frac{2x^3-x^2-1}{(x-1)(x+3)} \text{ is cont. everywhere except at } 1 \text{ and } -3.
\]

Exp. \(f(x) = \begin{cases} \frac{x-1}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases} = \begin{cases} x+1 & x \neq 1 \\ 1 & x = 1 \end{cases}
\)

is not continuous at 1
\[
\frac{1}{c} \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x-1}{x-1} = \lim_{x \to 1} x+1 = 2
\]
while \(f(1) = 1 \neq 2 \).

Exp. \(f(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases} = \begin{cases} x+1 & x \neq 1 \\ 2 & x = 1 \end{cases}
\)
is continuous at 1
\[
\frac{1}{c} \lim_{x \to 1} f(x) = 2 \text{ (as above)}
\]
and \(f(1) = 2 \).

[In fact, this function is the same as \(x+1\), since it is \(x+1\) for all \(x\).]

Exp. \(f(x) = \begin{cases} \frac{\sqrt{x^2+9}-3}{x} & x \neq 0 \\ \frac{1}{6} & x = 0 \end{cases}
\)
is continuous at 0