Note:

The final exam covers everything we did this semester including today's topics, 6.5 (average value of a function).

Check for Review Problems 5. Please do these problems before Wednesday class so you will understand better when I go over them.
Average value of a function.

The average of 2, 4 is \( \frac{2+4}{2} = 3 \).

Generally, the average of \( n \) numbers \( x_1, x_2, \ldots, x_n \) is
\[
\frac{x_1 + x_2 + \cdots + x_n}{n}
\]

Now, what is the average value of a function \( f(x) \) on an interval \([a, b]\)? First, \( f(x) \) has infinitely many values on \([a, b]\); so how can we talk about average of an infinitely many values? Here is how we can do it: Divide \([a, b]\) into \( n \) subintervals of equal length \( a < x_1 < x_2 < \cdots < x_n = b \), so that each subinterval has
length \( \Delta x = \frac{b-a}{n} \). Then we define the average of \( f \) over \([a, b]\) as the limit
\[
\lim_{n \to \infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}
\]
But this limit is just \( \frac{1}{b-a} \int_a^b f(x) \, dx \). Why?

\[
b/c \quad \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{1}{n} \left[ f(x_1) + f(x_2) + \cdots + f(x_n) \right] = \frac{1}{b-a} \cdot \frac{b-a}{n} \left[ f(x_1) + f(x_2) + \cdots + f(x_n) \right]
\]
\[
= \frac{1}{b-a} \left[ f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x \right]
\]
and
\[
\lim_{n \to \infty} \left[ f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x \right] = \int_a^b f(x) \, dx.
\]
Ex. Find the average value of \( x^2 \) over the interval \([0, 2]\).

Sol. \[
\frac{1}{2-0} \int_0^2 x^2 \, dx = \frac{1}{2} \left[ \frac{x^3}{3} \right]_0^2 = \frac{8}{2 \cdot 3} = \frac{4}{3} \] (Ans.)

Ex. Find the average value of \( \sin x \) over the interval \([0, \pi]\).

Sol. \[
\frac{1}{\pi-0} \int_0^\pi \sin x \, dx = \frac{1}{\pi} \left[ \cos x \right]_0^\pi
= \frac{1}{\pi} \left[ -\cos \pi + \cos 0 \right]
= \frac{1}{\pi} \left[ (1) - (-1) \right] = \frac{2}{\pi}.
\]

Ex. Find the average value of \( \frac{1}{3} x^2 - 1 \) over the interval \([-1, 2]\).

Sol. \[
\frac{1}{2-(-1)} \int_{-1}^2 \left( \frac{1}{3} x^2 - 1 \right) \, dx
= \frac{1}{3} \left[ \frac{x^3}{3} - x \right]_{-1}^2
= \frac{1}{3} \left[ \frac{8}{3} - 2 - \left( -\frac{1}{3} + 1 \right) \right]
= \frac{1}{3} (\frac{8}{3} - 2 + \frac{1}{3} - 1)
= \frac{1}{3} (3 - 3) = 0
\]