4. Use logarithmic diff. to find \( \frac{dy}{dx} \) \( \sqrt{\frac{3x+2}{3x-2}} \).

So\( \sqrt{\frac{3x+2}{3x-2}} \)

\[ \ln y = \frac{1}{2} \ln \frac{3x+2}{3x-2} = \frac{1}{2} \left[ \ln (3x+2) - \ln (3x-2) \right] \]

\[ \frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[ \frac{3}{3x+2} - \frac{3}{3x-2} \right] = \frac{3}{2} \left[ \frac{1}{3x+2} - \frac{1}{3x-2} \right] \]

\[ = \frac{3}{2} \cdot \frac{3x-2-(3x+2)}{(3x+2)(3x-2)} \]

\[ = \frac{3}{2} \cdot \frac{-4}{9x^2-4} \]

\[ = -\frac{6}{9x^2-4} \]

\[ \therefore \frac{dy}{dx} = y \left( -\frac{6}{9x^2-4} \right) \]

\[ = -\frac{6}{9x^2-4} \sqrt{\frac{3x+2}{3x-2}} \quad \text{(Ans.)} \]

**HW** 3.10 #17, #30 (Due Next class).

Solved problem 3.7 #39

and problem 3.10 #11 in class.
Linear Approximations and Differentials

Recall that a linear function is a function whose graph is a line. So every linear function can be written in the form \( ax + b \) where \( a \) and \( b \) are real constants.

Since linear functions are the SIMPLEST functions in math, it would be nice if we can approximate a complicated function by a linear function. How? Use the tangent line — the tangent line defines a linear function.

\[ y = f'(a)(x-a) \]

Note that \( f \) is not exactly equal to \( \frac{f(x) - f(a)}{x-a} \) for \( x \) near \( a \), but approximately equal to

\[ f(x) \approx f(a) + f'(a)(x-a) \]
3x. Find the linear approx. of \( x^2 \) at \( a = 3 \).

\[
\begin{align*}
\text{Svl} & \quad f(x) = x^2 \quad f(3) = 3^2 = 9 \\
& \quad f'(x) = 2x \quad f'(3) = 2 \cdot 3 = 6
\end{align*}
\]

\( f(3) + f'(3) (x-3) = 9 + 6 (x-3) = 6x - 9 \quad \text{(Ans.)} \)

\( x^2 \approx 6x - 9 \quad \text{for } x \text{ near 3.} \)

**Check:**

\[
\begin{align*}
3.1^2 & = 9.61 \\
6 \cdot 3.1 - 9 & = 18.6 - 9 = 9.6 \quad \frac{2}{3} \text{ close!}
\end{align*}
\]

\[
\begin{align*}
2.9^2 & = 8.41 \\
6 \cdot 2.9 - 9 & = 8.4 \quad \frac{7}{3} \text{ close!}
\end{align*}
\]

3x. Find the linear approximation of the function \( g(x) = \sqrt[3]{1+x} \) at \( a = 0 \) and use it to approximate the numbers \( \sqrt[3]{0.95} \) and \( \sqrt[3]{1.1} \).

\[
\begin{align*}
g(x) &= \sqrt[3]{1+x} \quad g(0) = 1 \\
g'(x) &= \frac{1}{3} (1+x)^{-\frac{2}{3}} \cdot 1 = \frac{1}{3} (1+x)^{-\frac{2}{3}} = \frac{1}{3 (1+x)^{\frac{2}{3}}} \quad , \quad g'(0) = \frac{1}{3} = \frac{1}{3}
\end{align*}
\]

\( g(0) + g'(0) (x-0) = 1 + \frac{1}{3} x \quad \text{(Ans.)} \)

\( \sqrt[3]{1+x} \approx 1 + \frac{1}{3} x \quad \text{for } x \text{ near 0} \)

\[
\begin{align*}
\sqrt[3]{0.95} &= \sqrt[3]{1-0.05} \\
&= 1 + \frac{1}{3} (-0.05) = 1 - \frac{0.05}{3} \approx 1 - 0.017 = 0.983
\end{align*}
\]
\( \sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \quad 0.1 \text{ is near 0} \)

\[ = 1 + \frac{1}{3}(0.1) = 1 + \frac{0.1}{3} \approx 1 + 0.033 = 1.033. \]

(Actual Ans. \( \sqrt[3]{1.1} = 1.03228 \ldots \); not too bad!)

Ex. Show that \( \sin x \approx x \) for \( x \) near 0.

So,

\[
\begin{align*}
\sin x & \approx f(0) + f'(0)(x - 0) = 0 + 1 \cdot x = x \\
\sin x & \approx x \quad \text{for } x \text{ near 0.}
\end{align*}
\]

Recall that

\[
f(x) = f(a) + f'(a)(x - a) \quad \text{for } x \text{ near a}
\]

Now let \( x = a + \Delta x \) where \( \Delta x \) is near 0 (so that \( x \) is near \( a \)), then

\[
f(a + \Delta x) \approx f(a) + f'(a)(a + \Delta x - a)
\]

\[
= f(a) + f'(a) \Delta x
\]

i.e., \( f(a + \Delta x) - f(a) \approx f'(a) \Delta x \quad (1) \)

Since \( a \) is arbitrary, (1) can be written as
\[ f(x + \Delta x) - f(x) \approx f'(x) \Delta x \quad \text{if } \Delta x \text{ is near 0} \]

Now let \[ \Delta y = f(x + \Delta x) - f(x), \]

and define \[ \Delta x = \Delta x, \quad dy = f'(x) \Delta x, \]

then \[ \Delta y \approx dy \]

If \( y = f(x) \), \( \Delta y \) is the actual change in \( y \) as \( x \) changes from \( x \) to \( x + \Delta x \); while \( dy = f'(x) \Delta x = f'(x) \Delta x \)

is an approximation of this change.

\[ \text{By. Let } y = x^2 + 3x \text{. Compare } \Delta y \text{ and } dy \quad \text{(a) for general } x \text{ and } \Delta x, \quad \text{(b) for } x = 2, \Delta x = 0.03. \]

\[ \text{Sol.} \]

\[ y = x^2 + 3x = f(x) \]

\[ \Delta y = f(x + \Delta x) - f(x) = (x + \Delta x)^2 + 3(x + \Delta x) - (x^2 + 3x) \]
\[ = x^2 + 2x \Delta x + \Delta x^2 + 3x + 3 \Delta x - x^2 - 3x \]
\[ = (2x + 3) \Delta x + \Delta x^2 \]

\[ dy = f'(x) \Delta x = (2x + 3) \Delta x = (2x + 3) \Delta x \]

So \[ \Delta y - dy = \Delta x^2 \quad \text{i.e.} \]

\[ \Delta y - dy \text{ are different by } \Delta x^2 \quad \text{(Ans.)} \]

This is very small when \( \Delta x \) is small.
(b) for \( x = 2 \), \( \Delta x = 0.03 \), we

\[
\Delta y = \Delta x = 0.03^2 = 0.0009 \quad \text{from part (a)}.
\]

OR (if you prefer not to use part (a))

\[
\Delta y = f(2 + 0.03) - f(2)
\]

\[
= f(2.03) - f(2)
\]

\[
= 2.03^2 + 3 \times 2.03 - (4 + 6)
\]

\[
= 4.1209 + 6.09 - 10 = 10.2109 - 10 = 0.2109
\]

\[
dy = f'(2) \, dx = f'(2) \Delta x = f'(2) \times 0.03
\]

\[
= (2 \times 2 + 3) \times 0.03 \quad \text{by } f'(x) = 2x + 3
\]

\[
= 7 \times 0.03
\]

\[
= 0.21
\]

\[
\therefore \, \Delta y - dy = 0.2109 - 0.21 = 0.0009
\]

3x. Use differentials to find an approximate value for (a) \((1.97)^6\)

(b) \(\cos 31.5^\circ\).

\[
\text{For } f(x) = x^6, \quad f'(x) = 6x^5
\]

\[
f(2) = 2^6 = 64 \quad f'(2) = 6 \times 2^5 = 6 \times 32 = 192
\]

\[
\Delta y = f(1.97) - f(2) = f(1.97) - 64
\]

\[
dy = f'(2) \, dx = 192 \times (1.97 - 2) = 192 \times (-0.03) = -5.76
\]

\[
\Delta y \approx dy
\]

\[
\therefore f(1.97) - 64 \approx -5.76
\]

\[
\text{ie. } f(1.97) = 64 - 5.76 = 58.24 \quad \text{(Ans.)}
\]

\[
\text{[Actual ans. by calculator } 1.97^6 = 58.451728 \text{]}
\]
\( (b) \) \quad f(x) = \cos x \quad f'(x) = -\sin x \quad \text{[Note: } x \text{ is in radian in this formula]}

31.5^\circ \text{ is close to } 30^\circ = \frac{\pi}{6} \text{ rad.}

\[ dx = \Delta x = 1.5^\circ \text{ in radian } = 1.5 \times \frac{\pi}{180} = \frac{\pi}{120} \]

\[ \Delta y = \cos 31.5^\circ - \cos 30^\circ = \cos 31.5^\circ - \frac{\sqrt{3}}{2} \]

\[ dy = f'(\frac{\pi}{6}) \, dx = -(\sin \frac{\pi}{6}) \times \frac{\pi}{120} = -\frac{1}{2} \times \frac{\pi}{120} = -\frac{\pi}{240} \]

\[ dy \approx \Delta y \]

\[ \cos 31.5^\circ - \frac{\sqrt{3}}{2} \approx -\frac{\pi}{240} \]

\[ \cos 31.5^\circ \approx \frac{\sqrt{3}}{2} - \frac{\pi}{240} = 0.8529 \]

[Actual value from calculator \( \cos 31.5^\circ = 0.85264 \ldots \)]

\[ \text{Q. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.} \]

\[ \text{Sol. volume of a hemisphere with radius } \]

\[ r \quad \text{is} \quad \frac{1}{2} \times \frac{4}{3} \pi r^3 = \frac{2}{3} \pi r^3 \]

\[ V = \frac{2}{3} \pi r^3 \]

\[ dV = \frac{2}{3} \pi r^2 \, dr = 2 \pi r^2 \, dr \]

\[ = 2 \pi \left( \frac{25 \times 100}{2} \right) (0.05) \]

\[ = 19634.95 \text{ cm}^3 \]

\[ \approx 1.96 \text{ m}^3 \]
8x. The radius of a circular disk is given as 2.4 cm with a maximum error in measurement of 0.2 cm.

(a) Use differentials to estimate the maximum error in the calculated area of the disk.

(b) What is the relative error?

Sol. \[ A = \pi r^2 \]

\[ dA = 2\pi r \, dr \]

\[ |dA| = 2\pi r |dr| \leq 2\pi \cdot 2.4 \cdot 0.2 = 9.6 \pi \text{ cm}^2 \text{ (Ans.)} \]

Relative error \[ = \frac{|dA|}{A} = \frac{9.6 \pi}{\pi \cdot 2.4^2} = \frac{9.6}{24^2} \approx 0.0167 = 1.67\% \]