3x. Find \( \frac{d}{dx} \ln(x^2 + 1) \)

**Sol.**

\[
\begin{align*}
\ln u & \rightarrow \frac{1}{u} \\
u = x^2 + 1 & \rightarrow 2x
\end{align*}
\]

\[
\begin{align*}
\frac{2x}{u} &= \frac{2x}{x^2 + 1} \\
(\text{Ans.})
\end{align*}
\]

Ex. Find \( \frac{d}{dx} \ln(g(x)) \)

\[
\begin{align*}
\ln u & \rightarrow \frac{1}{u} \\
u = g(x) & \rightarrow g'(x) \\
u = g(x) & \rightarrow g'(x)
\end{align*}
\]

So we have the formula:

\[
\begin{align*}
\frac{d}{dx} \ln(g(x)) &= \frac{g'(x)}{g(x)}
\end{align*}
\]

Ex. Find \( \frac{d}{dx} \ln(\sin x) \)

By above formula

\[
\frac{d}{dx} \ln(\sin x) = \frac{(\sin x)'}{\sin x} = \frac{\cos x}{\sin x} = \cot x \quad (\text{Ans.})
\]

Ex. Find \( \frac{d}{dx} \ln(\sqrt{x^2 + 1}) \)

Before we apply the formula, it is better to simplify \( \ln(\sqrt{x^2 + 1}) \) to \( \frac{1}{2} \ln(x^2 + 1) \) \[ \frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1} \]
\[ \frac{b/c.}{\text{(x}^2+1)^{\prime}\text{ is easy}} \]

while \((\sqrt{x^2+1})^{\prime}\text{ is hard!}\)

So

\[ \frac{d}{dx} \ln \sqrt{x^2+1} = \frac{d}{dx} \frac{1}{2} \ln (x^2+1) \]

\[ = \frac{1}{2} \cdot \frac{d}{dx} \ln (x^2+1) \]

\[ = \frac{1}{2} \cdot \frac{(x^2+1)^\prime}{x^2+1} = \frac{1}{2} \cdot \frac{2x}{x^2+1} = \frac{x}{x^2+1} \quad \text{(Ans.)} \]

\[ \text{Ex.}\quad \text{Find} \quad \frac{d}{dx} \ln \frac{x^5(2+x^2)^{3}}{\sqrt{x^2+x+1}} \]

Again, it is better to simplify the function first before applying the formula.

\[ \frac{b/c.}{\ln \frac{a}{b} = \ln a - \ln b} \]

\[ \ln \frac{x^5(2+x^2)^{3}}{\sqrt{x^2+x+1}} = \ln x^5 + \ln (2+x^2)^3 - \frac{1}{2} \ln (x^2+x+1) \]

\[ \frac{b/c.}{\ln a \cdot b = \ln a + \ln b} \quad \text{and} \quad \ln \frac{a}{b} = \frac{\ln a}{\ln b} \]

\[ \ln a^\frac{1}{2} = \frac{1}{2} \ln a \]

\[ \ln a^n = n \ln a \]

\[ \therefore \quad \frac{d}{dx} \ln \frac{x^5(2+x^2)^{3}}{\sqrt{x^2+x+1}} = 5 \cdot \frac{1}{x} + 3 \cdot \frac{2x}{2+x^2} - \frac{1}{2} \cdot \frac{2x+1}{x^2+x+1} \quad \text{(Ans.)} \]
30. Find \( \frac{d}{dx} \sqrt{\ln x} \).

**Sol.** Note: \( \sqrt{\ln x} \neq \frac{1}{2} \ln x \) \( \ln(x^{\frac{1}{2}}) = \frac{1}{2} \ln x \)

\[
\frac{d}{dx} \sqrt{\ln x} = \frac{d}{dx} \left( \ln x \right)^{\frac{1}{2}} = \frac{1}{2} \left( \ln x \right)^{\frac{1}{2} - 1} \cdot \frac{d}{dx} \ln x
\]

General power rule

\[
= \frac{1}{2} \left( \ln x \right)^{-\frac{1}{2}} \cdot \frac{1}{x}
\]

\[
= \frac{1}{2x \sqrt{\ln x}} \quad (\text{Ans.})
\]

Consider the function \( \ln |x| \).

Graph of \( \ln |x| \)

\[\ln |x| = \begin{cases} 
\ln x & \text{if } x > 0 \\
\ln(-x) & \text{if } x < 0
\end{cases}\]

\[
\Rightarrow \quad \frac{d}{dx} \ln |x| = \frac{d}{dx} \ln x = \frac{1}{x} \quad \text{if } x > 0
\]

By formula

\[
= \frac{d}{dx} \ln(-x) = \frac{(x')'}{-x} = \frac{-1}{-x} = \frac{1}{x} \quad \text{if } x < 0
\]

So you see that

\[
\frac{d}{dx} \ln |x| = \frac{1}{x} \quad \text{no matter whether } x > 0 \text{ or } x < 0.
\]

\[
\left\{ \frac{d}{dx} \ln |x| = \frac{1}{x} \right\}
\]
Note: \( \ln x \) is defined only for \( x > 0 \)

while \( \ln |x| \) is defined for all \( x \) except \( x = 0 \).

\[ \text{3x. Find } \frac{d}{dx} \ln |x^2-x-1| \]

\[
\begin{align*}
\text{Sol.:} & \quad \ln |u| \left( \frac{d}{du} \ln \frac{1}{u} \right) \\
& \quad u = x^2-x+1 \quad \frac{d}{dx} 2x-1 \\
& \quad = \frac{2x-1}{x^2-x-1} \quad (\text{Ans.})
\end{align*}
\]

Generally, we have

\[
\left\{ \frac{d}{dx} \ln |g(x)| = \frac{g'(x)}{g(x)} \right\}
\]

Note: Sometimes the absolute value is unnecessary.

For example \( \ln |x^2+1| = \ln (x^2+1) \)

\[ \text{abs. val. not necessary } \ln x^2+1 > 0 \]

while \( \ln |x-1| \neq \ln (x-1) \)

\[ \text{abs. val. } x-1 \text{ can be negative.} \]
Suppose you have a complicated function like
\[
\frac{x^{3/4} \sqrt{x+1}}{(3x+2)^5}
\]
and you need to find its derivative.

You can apply quotient rule combined with general power rule & product rule but that is too hard.

One way out is the following:

Let
\[
y = \frac{x^{3/4} \sqrt{x+1}}{(3x+2)^5}
\]

Then
\[
\ln y = \ln x^{3/4} \sqrt{x+1} - \ln (3x+2)^5
\]
\[
\ln y = \ln x^{3/4} + \ln \sqrt{x+1} - 5 \ln (3x+2)
\]
\[
\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln (x+1) - 5 \ln (3x+2)
\]

Then taking implicit diff: 
\[
\frac{1}{y} \cdot \frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x+1} - 5 \cdot \frac{3}{3x+2}
\]
\[
= \frac{3}{4x} + \frac{x}{x+1} - \frac{15}{3x+2}
\]

So
\[
\frac{dy}{dx} = y \left( \frac{3}{4x} + \frac{x}{x+1} - \frac{15}{3x+2} \right)
\]
\[
= \frac{x^{3/4} \sqrt{x+1}}{(3x+2)^5} \left( \frac{3}{4x} + \frac{x}{x+1} - \frac{15}{3x+2} \right) \quad \text{(Ans.)}
\]

This method is called logarithmic differentiation.
HW (due next class): Use logarithmic differentiation to find \[ \frac{d}{dx} \sqrt[3]{\frac{3x+2}{3x-2}}. \]

Remember, the power rule \( \frac{d}{dx} x^n = nx^{n-1} \). We only proved it for some special \( n \)'s. Now we can prove it for any real number \( n \):

\[ y = x^n \]

Take absolute value of both sides: \[ |y| = |x|^n \]

Take \( \ln \) of both sides: \[ \ln|y| = \ln|x|^n \quad \left[ \frac{\ln a^n}{a} = n \ln a \right] \]

\[ \ln|y| = n \ln|x| \]

Take their implicit diff: \[ \frac{1}{y} \frac{dy}{dx} = n \frac{1}{x} \]

\[ \frac{dy}{dx} = nx \cdot \frac{1}{x} = n x^{n-1} \]

2x. Find \( \frac{d}{dx} x^x \)

Note that you can't apply power rule since the exponent is not a constant.

Also, you can't apply \( \frac{d}{dx} a^x = a^x \ln a \) formula.

6c. \( x^x \)

is not a constant.
So let \( y = x^x \). Then \( \ln y = \ln x^x = x \ln x \)

**Implicit diff:** \( \frac{1}{y} \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} \) (product rule)

\[ = \ln x + 1 \]
\[ = 1 + \ln x \]

\[ \therefore \frac{dy}{dx} = y \left( 1 + \ln x \right) = x^x \left( 1 + \ln x \right) \]

(Ans.)

**Another way:**

\[ x^x = e^{\ln x^x} = e^{x \ln x} \]

\[ e^u \frac{du}{dx} = e^u \]

\[ u = x \ln x \]

\[ \frac{du}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 = 1 + \ln x \]

\[ \therefore \text{Ans.} = e^u \left( 1 + \ln x \right) \]
\[ = e^{x \ln x} \left( 1 + \ln x \right) \]
\[ = x^x \left( 1 + \ln x \right) \]
Related Rates Problems

Ex. 1. If a snowball melts so that its surface area decreases at a rate of 1 cm²/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Sol. Write \( S \) for surface area and \( D \) for diameter.

We have

\[
S = 4\pi r^2 = 4\pi \left(\frac{D}{2}\right)^2 = 4\pi \cdot \frac{D^2}{4} = \pi D^2
\]

Surface area decreases at a rate of 1 cm²/min

\[
\Rightarrow \quad \frac{dS}{dt} = -1 \quad \text{(constant rate)} \quad t = \text{time in minutes}
\]

From \( S = \pi D^2 \)

we have, taking \( \frac{d}{dt} \) of both sides

\[
\frac{dS}{dt} = \pi \cdot 2D \frac{dD}{dt}
\]

\[
\therefore \quad \frac{dD}{dt} = \frac{1}{2\pi D} \frac{dS}{dt}
\]

\[
= \frac{1}{2\pi \times 10} = \frac{1}{20\pi} \approx -0.06 \text{ cm/min. (Ans.)}
\]

[Note that \( D \) is not decreasing at a constant rate!]

When \( D \) becomes small, the rate of decreasing rate is much higher]
A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is his shadow on the building decreasing when he is 4 m from the building?

Sol. Let \( x \) be the distance from the lamp light to the man. Let \( y \) be the length of his shadow. Then

\[
x \cdot y = 24
\]

Taking derivative w.r.t. \( t \) [remember that \( x \) and \( y \) are functions of \( t \)],

\[
x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt} = 0
\]

\[
\frac{dy}{dt} = -\frac{y}{x} \cdot \frac{dx}{dt}
\]

Now \( \frac{dx}{dt} = 1.6 \) and when he is 4 m from the building

\[
x = 8, \quad y = \frac{24}{x} = \frac{24}{8} = 3, \quad \text{so}
\]

\[
\frac{dy}{dt} = -\frac{3}{8} \cdot 1.6 = -0.6 \text{ m/s.} \quad \text{(Ans.)}
\]
2x. 3 At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 pm?

Sol. Let \( D = \) distance between ships

\[
D = (150 - x)^2 + y^2
\]

\[
\frac{dx}{dt} = 35
\]

\[
\frac{dy}{dt} = 25
\]

\[
2D \frac{dD}{dt} = 2(150 - x) \left( \frac{dx}{dt} \right) + 2y \frac{dy}{dt}
\]

\[
\left( \frac{dD}{dt} \right)^2 = (150 - x)^2 \left( \frac{dx}{dt} \right)^2 + y^2 \frac{dy}{dt}^2
\]

At 4:00 pm, \( x = 35 \times 4 = 140, \ y = 25 \times 4 = 100 \)

\[
D = (150 - x)^2 + y^2 = (150 - 140)^2 + 100^2
\]

\[
= 100 + 10000 = 10100
\]

\[D = \sqrt{10100}\]

\[
\therefore \text{From (1)}
\]

\[
\sqrt{10100} \frac{dD}{dt} = 10(-35) + 100 \cdot 25
\]

\[
= 2500 - 350 = 2150
\]

\[
\therefore \frac{dD}{dt} = \frac{2150}{\sqrt{10100}} \approx 21.4 \text{ km/h}
\]
Ex. 4. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

\[ s = 1 + x^2 \]

\[ \frac{ds}{dt} = -1 \]

Find \( \frac{dx}{dt} \) when \( x = 8 \)

\[ 2s \frac{ds}{dt} = 2x \frac{dx}{dt} \]

When \( s = x = 8 \),

\[ s = 1 + x^2 = 1 + 64 = 65 \]

\[ \therefore s = \sqrt{65} \]

\[ s \frac{ds}{dt} = x \frac{dx}{dt} \]

\[ \frac{65}{\sqrt{65}} \frac{dx}{dt} = 8 \cdot \frac{dx}{dt} \]

\[ \Rightarrow \frac{dx}{dt} = -\frac{\sqrt{65}}{8} = -1.008 \text{ m/s} \]
Ex. 5. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?

\[ \cos^2 \theta = \frac{x}{100} \]

\[-\csc \theta \frac{d\theta}{dt} = \frac{1}{100} \frac{dx}{dt} \quad - (1)\]

At the moment when \( s = 200 \), \( x = \sqrt{200^2 - 100^2} = \sqrt{30000} = 100 \sqrt{3} \)

\[ \csc \theta = \frac{s}{100} = \frac{200}{100} = 2 \]

\[ (1) \implies -4 \frac{d\theta}{dt} = \frac{1}{100} \cdot 8 \]

\[ \frac{d\theta}{dt} = -\frac{2}{100} = -\frac{1}{50} = -0.02 \text{ rad/s} \]