The Many Facets of Chaos

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Definitions of Chaos

“Chaos” ala Google

• Typical dynamical systems either have simple trajectories such as steady state and quasiperiodic orbits, or they exhibit chaos.

• Chaos is defined in so many ways that it is quite confusing for a practitioner to get a reasonable answer to the simple question

  “What is the definition of chaos?”

• We assert that this is the wrong question to ask
Definitions of Chaos

Buddist parable of the three blind monks and the elephant

- Chaos cannot be satisfactorily defined mathematically using a single definition, *not* because chaos is not a single concept, but because chaos has many manifestations in many different situations.

- In this talk: A variety of manifestations of chaos, with the conjecture that *typically* the different forms of chaos are equivalent.

- Disclaimer: While this illustrates a universal point, the specific list is incomplete, shaped by our personal knowledge and experience
Earliest Observations: Transverse Homoclinic Orbits

Stable and unstable manifolds of fixed point, Hénon map

\[(x, y) \mapsto (\rho - x^2 - 0.3y, x), \quad -4 < x < 4, \quad -3 < y < 3, \quad \rho = 2.0, 2.01725, 2.01875, 2.0246\]

- Poincaré initially thought all homoclinic orbits coincided when they intersected.
- Phragmén pointed out his error.
Poincaré, 1892: “If one seeks to visualize the pattern formed by these two curves and their infinite number of intersections, each corresponding to a doubly asymptotic solution, these intersections form a kind of lattice-work, a weave, a chain-link network of infinitely fine mesh; each of the two curves can never cross itself, but it must fold back on itself in a very complicated way so as to recross all the chain-links an infinite number of times. One will be struck by the complexity of this figure, which I am not even attempting to draw.”

Movie
Smale, 1967: Horseshoe maps are contained in transverse homoclinic orbits, implying chaos.

Useful characterization when visualization is difficult but analysis is tractable, such as delay equations, PDEs.

Often gives rise to transient chaos, not attracting

Not quantitative
Ueda 1961, Lorenz 1963:
Robustness and Irregular Topology

Ueda-Duffing map: \( x''(t) + 0.05x'(t) + x(t)^3 = 7.5 \sin(t), \)
\[-2.2 < x < 2.2, -1.5 < y < 2.6, \text{ 2}\pi\text{- stroboscopic}\]

Right, stable manifold branches

Lorenz made similar observations
An eyeball measure of fractal topology in an attractor of a low-dimensional system is an easy method of chaos detection.

This has since been made more precise in the form of attractor dimension calculations.

There is no clear definition of strange attractors – rank one attractors are one special subtype [Wang and Young]

In fact, homoclinic points play a role...
Holmes map: \((x, y) \mapsto (1.5x + x^3 + \lambda y, x)\),
\(-2 < x < 2, -2 < y < 2, \lambda = .8 \text{ (up)}, .9 \text{ (down)}\).
Gollub and Swinney 1975 observed chaotic motion in Taylor-Couette flow fluid experiments. 

- No underlying map
- Data in the form of a time series
- Indicators were based on the broad power spectrum for the time series data
- This method only considers behavior of orbits, ignoring nearby trajectories
Exponential Divergence of Trajectories

\[ x \mapsto 2x \mod 1 \]

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- Exponential divergence of trajectories used in definition of scrambled sets
- Generalizes to positive stretching along solutions, meaning positive Lyapunov exponent
- Lyapunov chaos: Positive probability of a random trajectory having an expanding direction.
Lyapunov Chaos

- Common method of checking for chaos for maps and flows
- Gives quantitative measure of degree of chaos
- Finite time Lyapunov exponents are used for time series data using delay coordinate embeddings and attractor reconstruction
- In cases of noisy data, a full attractor reconstruction may give less reliable results than a two-dimensional projection. [Mytowicz, Diwan, Bradley, Computer cache data, 09]
- Spurious Lyapunov exponents can occur in time series, with no easy way to distinguish the true exponents [Sauer, Tempkin, Yorke, 98]
- Methods such as the 0 – 1 test attempt to avoid reconstruction [Gottwald, Melbourne]
For map $f$, let $S_p$ be the number of fixed points of $f^p$. Periodic orbit chaos means positive periodic orbit entropy:

$$\lim_{p \to \infty} \sup \frac{\log S_p}{p}$$

We used periodic orbit chaos to show results on period-doubling cascades [Sander and Yorke, 09–13]

Without theoretical methods, computation cannot be made rigorous
Positive Topological Entropy

- Topological entropy measures the mixing of a set. For $N(n, \varepsilon)$ the distinguishable orbits, topological entropy:
  \[
  \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \frac{\log N(n, \varepsilon)}{n}
  \]

- Positive topological entropy is a concept of chaos useful in the case of continuous, analytically defined, not necessarily smooth maps.

- It is not easily numerically computable, though has been used for rigorous computational proofs [Day et al. and Newhouse et al. 08, Frongillo 14]
Comparing to Entropies

- Metric entropy is related: Topological entropy is an upper bound on metric entropy.
- In finite dimensional maps and flows, topological entropy for smooth maps is equal to the sum of the positive Lyapunov exponents when a measure is SRB. Otherwise, the difference is in terms of the dimension of the invariant measure [Pesin, Ruelle, Margulis, Ledrappier/Young 85].
- The relationship is unknown for general infinite dimensional equations and time series.
Comparing Entropies

- **Periodic orbit chaos** is equivalent to **positive topological entropy** for Axiom A diffeomorphisms [Bowen, 70] and Hénon-like maps [Wang and Young]. There can be superexponential growth of periodic orbits [Kaloshin]. Thus the concepts are in general not equivalent.

- There are zero topological entropy sets which are scrambled sets.

- No clear relationship between positive Lyapunov exponents and decay of time correlations [Slipantschuk, Bandtlow, Just, 13].
Chaotic Saddles, Robust But Not Stable

Holmes map: \((x, y) \mapsto (1.5x + x^3 + \lambda y, x),\)

\(-2 < x < 2, -2 < y < 2, \lambda = 0.8, 0.95\)
Fractal Basins

Forced-damped pendulum:
\[ x'' + 0.2x' + \sin x = \rho \cos t, \]
\[-\pi < x < \pi, 2 < y < 4, \]
\[ \rho = 1.5725, 1.73, 2.3225, 3.0875, 2\pi - \text{stroboscopic} \]

- Attractors globally attracting (3), multiple basins (1,2,4)
- Fractal basin boundaries (1,2)
- Eight distinct basins (2)
- Chaotic attractors (3,4)

Movie
Two Conjectures

What happens typically? Typical means either a generic or prevalent set.

**Chaos Conjecture:** For a typical smooth dynamical system, the definitions of chaos are equivalent: three types of entropy, positive Lyapunov exponent, transverse homoclinic orbits, horseshoes.

**Typical Behavior Conjecture:** Consider a basic set: maximal compact invariant transitive set for a map or flow in finite dimensions. For a typical set of equations, if this set does not have positive Lyapunov exponent, then the set is a steady state, periodic orbit, or quasiperiodic set. Quasiperiodic means topological circle or torus of some dimension (for maps, multiple tori).
Current work: Quasiperiodicity

- **Fast accurate method** for calculating *Lyapunov exponent*, *rotation number*, *Fourier coefficients* for a quasiperiodic map.

- The method resembles *windowing methods* in signal processing.
Conjugacy in 1D and 2D
Fourier coefficients decay in 1 and 2 dimensions
Each definition of chaos comes with its own strengths and shortcomings – both numerical [Barrio, Borczyk, Breiter 07] and theoretical.

The concept is too big for a definition – no one mathematical definition will suffice.

“Scientists work by concepts rather than definitions ... Nature abhors a definition try to lock something into too small a box and I guarantee nature will find an exception.” - *Discover Magazine*, in reference to Pluto and the demise of its planethood.