TAYLOR POLYNOMIALS

Taylor polynomials are used to approximate a function near a particular input value. To illustrate the main idea, we will start with an example: approximating the exponential function $e^x$ near the location $x = 0$.

First, note that $e^x$ has value 1 at $x = 0$. Thus a first approximation, which we will call level 0, is the constant function $y = 1$ for all $x$.

To do better, look at $e^x - 1$ near $x = 0$. This is the numerator for the computation of the derivative of $e^x$ at $x = 0$ as a limit, namely

$$\lim_{x \to 0} \frac{e^x - 1}{x - 0} = 1. \tag{1}$$

This can be rewritten by subtracting 1 from both sides of the equation and then putting the constant 1 into the limit. This reads as follows:

$$\lim_{x \to 0} \frac{e^x - 1}{x - 0} - 1 = \lim_{x \to 0} \frac{e^x - 1 - x}{x - 0} = 0. \tag{2}$$

The numerator is the difference between the exponential and the tangent line approximation $y = 1 + x$ and it is so small that even dividing by $x$ and taking the limit still gives 0. This approximation, which we will denote by $f_1$, is $y = f_1(x) = 1 + x$ and will be our “level 1” approximation.

To go further, consider the numerator from the previous limit but with denominator now the smaller function $x^2$. Using L’Hospital’s Rule and the previous limit for the derivative to evaluate the new limit yields a very interesting result:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^2} = \frac{1}{2}. \tag{3}$$

If we again unravel this by subtracting and then rewriting the limit as a single limit of a fraction, this becomes

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} - \frac{1}{2} = \lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^2} = 0. \tag{4}$$

This says that the difference between the exponential and the quadratic function $y = f_2(x) = 1 + x + \frac{1}{2}x^2$ is so small that its limit, after dividing by $x^2$, is 0 as $x$ approaches 0. This will be our “level 2” approximation.

To generate the next approximation, look at the same numerator as our last calculation but change the denominator to $x^3$, which is smaller near $x = 0$. Either one step of L’Hospital followed by use of the previous result, or three applications of L’Hospital, which we don’t write down below, yields the interesting result:

$$\lim_{x \to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3} = \lim_{x \to 0} \frac{e^x - 1 - x}{3x^2} = \frac{1}{6}. \tag{5}$$

Once again moving the number to the other side and writing the new limit as a single fraction, now with denominator $x^3$, yields the cubic polynomial that is our
“level 3” approximation: \( y = f_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \). Its difference with the exponential is so small that in the limit, even after dividing by \( x^3 \), we get 0.

The polynomials that have been found are examples of Taylor polynomials. They provide good approximations near the single point \( x = 0 \) and each one is built on top of its predecessors by adding a new term involving a higher power of \( x \).

Now let’s ask a slightly different question: How can we approximate the same function, but near the point \( x = 1 \). Now the relevant value is \( e^x \) at \( x = 1 \), which is \( e \). To find the “level 1” approximation, look at the difference \( e^x - e \), which is the numerator of the fraction that comes in the definition of derivative of \( e^x \) at \( x = 1 \). The denominator of that fraction is the difference in inputs, namely \( x - 1 \). Since the derivative of \( e^x \) is again \( e^x \), the following equation involving limits holds, similar to equation (1) above:

\[
\lim_{x \to 1} \frac{e^x - e}{x - 1} = e. \tag{6}
\]

This provides the linear function, which is the tangent line approximation, namely \( y = e + e(x - 1) \). This is a good approximation near \( x = 1 \) but note that it is not very good near \( x = 0 \). In fact, it gives the value 0 at \( x = 0 \), which is pretty far off.

To continue the approximation near \( x = 1 \), subtract and then up the power in the denominator to \((x - 1)^2\) and then take the limit, using L’Hospital and the previous result:

\[
\lim_{x \to 1} \frac{e^x - e - e(x - 1)}{(x - 1)^2} = \lim_{x \to 1} \frac{e^x - e}{2(x - 1)} = \frac{1}{2} e. \tag{7}
\]

This gives the quadratic approximation (“level 2”) after subtraction and grouping as a single fraction. The outcome is: \( y = e + e(x - 1) + \frac{1}{2} e (x - 1)^2 \)

This is a closer approximation to the function near \( x = 1 \). This polynomial has the same value, first derivative, and second derivative as the exponential function at the location \( x = 1 \) by construction.

With these examples in hand, the general case is pretty similar.

**What are Taylor polynomials?**

For a general function \( f(x) \) with derivatives near a point \( x = a \) up to some order \( n \), we can define the Taylor polynomial of degree \( n \), denoted \( T_n(x) \), that approximates \( f \) near \( x = a \) as follows:

\[
T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n \tag{8}
\]

where the dots represent terms of similar type along the way. It is an interesting proof by induction or repeated use of L’Hospital’s rule that the following limit holds:

\[
\lim_{x \to a} \frac{f(x) - T_n(x)}{(x-a)^n} = 0. \tag{9}
\]

This again means that the difference is so small that the limit, even after dividing by the small function \((x - a)^n\), is 0. This is now a very general “recipe”
for approximation near one particular input value. Note also that the Taylor polynomials of different degree only differ by adding terms of higher powers in the difference $x - a$.

Taylor’s theorem is a fancier version of the above limit statement, in that it does not use limits but keeps $x$ fixed and different from $a$. It is a kind of mean value theorem, much like the mean value theorem for derivatives, which is the case when the degree of the polynomials, which is called $n$ above, is precisely 1.

**Another example** Find the Taylor polynomial of degree 2 that approximates $f(x) = \sqrt{1 - x}$ near $x = 0$.

The “recipe” uses the polynomial $f(0) + f'(0) x + \frac{1}{2} f''(0) x^2$. We compute the coefficients successively, using the chain rule:

$$f(0) = 1$$

$$f'(x) = (-1)\left(\frac{1}{2}(1 - x)^{-\frac{1}{2}}\right), \text{ so } f'(0) = -\frac{1}{2};$$

$$f''(x) = (-1)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(1 - x)^{-\frac{3}{2}}, \text{ so } f''(0) = -\frac{1}{4}. $$

This gives the answer: The Taylor polynomial of degree 2 near $x = 0$ for the function $\sqrt{1 - x}$ is: $1 - \frac{1}{2}x - \frac{1}{8}x^2$. This is a good approximation near $x = 0$. Notice that the square root is not defined beyond $x = 1$ but the polynomial is oblivious to that, so the moral again is:

**Taylor polynomials are tools to approximate functions near a particular location. They match the function value and as many derivatives as they can at that particular value.**

You will be graphing these approximations and working with them in the first computer lab assignment, to be given out shortly. This is an important calculus tool in later science, engineering, economics, and mathematics.