NAME

| Answer Key |
| Math 125-B01, Summer 2014, Test 3, O’Beirne |

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<thead>
<tr>
<th></th>
<th>a</th>
<th>4, 4</th>
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</table>
1. (a) Given the arithmetic sequence with first term 9 and common difference \(-\frac{2}{3}\).

What is the 42\textsuperscript{nd} term? \(\frac{-18}{3}\)

What is the sum of the first 27 terms? \(9\)

Work:
\[
9 + 41\left(-\frac{2}{3}\right) = -\frac{18}{3}
\]
\[
\frac{27}{2} \left( 2 \cdot 9 + 26\left(-\frac{2}{3}\right) \right) = 9
\]

(b) Given the geometric sequence with first term 1,000 and common ratio 1.05.

What is the 17\textsuperscript{th} term of the sequence? \(\frac{1000 \cdot (1.05)^{16}}{1.05} = 21,182.87\)

What is the sum of the first 20 terms? \(\frac{1000 \cdot \left[1 - (1.05)^{20}\right]}{1 - 1.05} = 33,045.95\)

Work:
\[
S_{17} = a \cdot r^{16} = 1000 \cdot (1.05)^{16}
\]
\[
S_{20} = \frac{a \cdot r^{20} - a}{1 - r} = \frac{1000 - 1000 \cdot (1.05)^{20}}{1 - 1.05} = \frac{1000 \cdot \left[1 - (1.05)^{20}\right]}{1 - 1.05}
\]
2. (a) Solve the following recurrence relation explicitly for \( a_n \)

\[ a_n = 3a_{n-1} - 2a_{n-2} \text{ with } a_0 = 1 \text{ and } a_1 = 3 \]

\[
a_n = \frac{2^{n+1} - 1}{3}
\]

What is \( a_{15} \) ? \( 2^{16} - 1 \approx 65,535 \) check

\[
\begin{align*}
\chi^2 - 3\chi + 2 &= 0 \\
(\chi - 1)(\chi - 2) &= 0 \\
\chi_1 &= 1 \\
\chi_2 &= 2 \\

a_n &= c_1 1^n + c_2 2^n
\end{align*}
\]

\[
\begin{align*}
n = 0 & : a_0 = 1 = c_1 + c_2 \quad \Rightarrow \quad c_1 = -1 \\
n = 1 & : a_1 = 3 = 2c_1 + 2c_2 \quad \Rightarrow \quad c_2 = 2
\end{align*}
\]

(b) Solve the following recurrence relation explicitly for \( a_n \)

\[ a_n = 4a_{n-1} - 4a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 6 \]

\[
a_n = \frac{2^{n+1} + 2^n}{3} = 2^{n+1} + 2^n
\]

What is \( a_{15} \) ? \( 2^{16} + 1 = 557,056 \) check

\[
\begin{align*}
\chi^2 - 4\chi + 4 &= 0 \\
(\chi - 2)^2 &= 0 \\
\chi &= 2 \\

a_n &= c_1 2^n + c_2 2^n = 2 \cdot 2^n + 1 \cdot 2^n = 2^{n+1} + 2^n = 2^{n+1} + 2^n
\end{align*}
\]

\[
\begin{align*}
m = 0 & : a_0 = 2 = c_1 + 0 \quad \Rightarrow \quad c_1 = 2 \\
m = 1 & : a_1 = 6 = 2c_1 + 2c_2 \quad \Rightarrow \quad c_2 = 1
\end{align*}
\]

Check: 2, 6, 16, 40
3. How many of the integers between 1 and 800 (inclusive)…
   (i) are divisible by 2 or 3 or 5?

   Work:

   \[
   \begin{align*}
   \left\lfloor \frac{800}{2} \right\rfloor &= 400 & \left\lfloor \frac{800}{6} \right\rfloor &= 133 \\
   \left\lfloor \frac{800}{3} \right\rfloor &= 266 & \left\lfloor \frac{800}{10} \right\rfloor &= 80 \\
   \left\lfloor \frac{800}{5} \right\rfloor &= 160 & \left\lfloor \frac{800}{15} \right\rfloor &= 53
   \end{align*}
   \]

   \[
   |A_2 \cup A_3 \cup A_5| = 400 + 266 + 160 - 133 - 80 - 53 + 26 = 586 \tag{Principle of Incl/Excl}
   \]

   (ii) are divisible by 5 but not by 2 or 3?

   From above:

   \[
   \begin{align*}
   2 & \quad 3 \\
   16 & \quad 27 & \quad \text{(Count these)}
   \end{align*}
   \]

   \[
   |A_5| = 53
   \]
4. (a) How many numbers in the range 100-799 (inclusive)...

(i) have no repeated digits?
Work: 

(ii) are odd and have no repeated digits?
Work:
Case 1: First digit odd
      \[4 \times 8 \times 4 = 128\]
Case 2: First digit even
      \[3 \times 8 \times 5 = 120\]
\[248\]

(iii) are even and have no repeated digits?
Work:
\[504 - 248 = 256\]

(b) How many 4-digit numbers are there from 1000 to 9999 (inclusive)

(i) if repetitions are allowed?
Work:
\[9 \times 10 \times 10 \times 10 = 9000\]
5. (a) The Pigeonhole Principle shows that in any sequence of 10 natural numbers there is a “string” of consecutive terms whose sum is divisible by 10. In the following sequence circle that string.

(ii) if repetitions are not allowed?
Work

\[
\begin{array}{c}
81 \\
56 \\
486 \\
405 \\
36 \\
\end{array}
\]

(iii) if one or more digits are repeated?
Work:

\[9000 - 4536 = 4464\]

(b) A is a set containing seven different natural numbers, each less than or equal to 21. Use the Pigeonhole Principle to show that if the elements in each non-empty subset of A are summed, the total for at least two of the non-empty subsets must be equal. (Hint: How many non-empty subsets are there?)
Work:

A has \(2^7 - 1\) non-empty subsets: \(2^7 - 1 = 127\.

Smallest non-empty set count is 1
\{1\}

Largest is 126
\{15, 16, ..., 21\}

Largest sum of 7 naturals
Each diff is \(\pm 21\) in
\(21 + 20 + 19 + 18 + 17 + 16 + 15 = 126\)

If you assign 127 sets to 126 numbers then at least 2 sets must have the same number (PP) (Pigeonhole Principle)
6. Suppose you take a test and you must answer exactly 9 of 12 questions.

(a) In how many ways can you choose the questions you answer? (The order doesn’t matter)

Work:
\[
\binom{12}{9} = \binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220
\]

(b) In how many ways can you choose 9 questions if you must answer all four of the first questions?

Work:
Choose 5 \& last 8
\[
\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56
\]

(c) In how many ways can you choose 9 if you must answer at least three of the last six questions and at most three of the first four questions?

Work:
6 Cases:
\[
\begin{align*}
\text{First} & \quad \text{Middle} & \quad \text{Last+} \\
3 & 2 & 4 & = 9 & \left(\binom{4}{2}\binom{5}{4}\right) & = \left(\binom{4}{2}\binom{6}{4}\right) = 4 \cdot 1 \cdot 15 = 60 \\
2 & 2 & 5 & = 9 & \left(\binom{3}{2}\binom{5}{5}\right) & = 6 \cdot 1 \cdot 6 = 36 \\
3 & 1 & 5 & = 9 & \left(\binom{3}{3}\binom{5}{5}\right) & = 1 \cdot 1 \cdot 5 = 5 \\
1 & 2 & 6 & = 9 & \left(\binom{3}{2}\binom{6}{6}\right) & = 4 \cdot 1 \cdot 1 = 4 \\
2 & 1 & 6 & = 9 & \left(\binom{4}{1}\binom{6}{6}\right) & = 6 \cdot 1 \cdot 1 = 6 \\
3 & 0 & 6 & = 9 & \left(\binom{4}{3}\binom{6}{6}\right) & = 4 \cdot 1 \cdot 1 = 4
\end{align*}
\]

\[
\{60, 36, 5, 4, 6, 4\} = 164
\]

(d) In how many ways can you choose 9 if you must answer at least four of the last six questions?

Work:
\[
\begin{align*}
\text{First} & \quad \text{Last+} \\
\frac{6}{5} & \quad \frac{6}{4} & \quad \frac{6}{3} \\
\frac{6}{5} & \quad \frac{4}{5} & \quad \frac{6}{3} \\
\binom{6}{4} & \binom{6}{5} & \binom{6}{6}
\end{align*}
\]

\[
\{60, 15, 20\} = 200
\]