1. Test the validity of the following argument. If it is a valid argument show why. If it is not a valid argument specify truth values for p, q, and r that produce true premises but a false conclusion.

\[
p \lor (q \rightarrow r) \\
q \lor r \\
r \rightarrow p \\
\hline
\neg p \lor q
\]

Work: \text{\textit{Invalid}}

Use Truth Table to see that when \( p = T, q = F, r = T \) the premises are \( T \) but conclusion is \( F \)

2. (a) If \( A \cup B = A \cup C \) does \( B = C \)? Explain fully. \( \text{No} \)

Work: \textit{Counterexample}

Let \( A = \{1,2\} \), \( B = \{1\} \), \( C = \{2\} \).

Then \( A \cup B = A \cup C \) but \( B \neq C \)

2. (b) If \( A \cap B = A \cap C \) and \( A^c \cap B = A^c \cap C \) does \( B = C \)? Explain fully. \( \text{Yes} \)

Work: \( B = (A \cap B) \cup (A^c \cap B) = (A \cap C) \cup (A^c \cap C) = C \) So \( B = C \)
3. Given the following tables which depict relations. For each one, circle whether the relation is Reflexive (R), Symmetric (S), Transitive (T), or Anti-Symmetric (A-S). Also circle whether each represents an Equivalence Relation (ER) or a Partial Order (PO).

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>X</td>
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<td></td>
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<tr>
<td>c</td>
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<tr>
<td>d</td>
<td>X</td>
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<td></td>
<td>d</td>
</tr>
</tbody>
</table>

R    | R    |
S    | S    |
T    | T    |
A-S  | A-S  |
ER   | ER   |
PO   | PO   |

4. (a) Express in disjunctive normal form: \( p \lor ((\neg p) \land q) \)

Work:

\[
\begin{align*}
(p \land \neg q) \lor (p \land q) \lor (\neg p \land q)
\end{align*}
\]

(b) Given that \(287^2 + 1 = 2(41,185)\) express 41,185 as the sum of two squares

Work:

\[
\begin{align*}
m^2 + 1 &= 2m \\
m^2 - 2m + 1 &= 2(4k + 1) \\
(2k+1)^2 + 1 &= 2m \\
k^2 + 2k + 1 &= 2m \\
m &= 4k^2 + 2k + 1
\end{align*}
\]

(c) Assuming \(k \leq d\) list the elements of the set \{dkk, d, kk, dk, ddk, kd, dkkd, kdk\} in lexicographic order.

kk, kd, kdk, d, dk, ddk, dkkd, ddk
5. (a) Proposition 2.4.3: Let \( \sim \) be an equivalence relation on \( A \). For any \( x \in A \), \( x \sim a \) if and only if sets \( \bar{x} = \bar{a} \).

Proof (fill in the blanks with the reason for that step):
First show that sets \( \bar{x} = \bar{a} \rightarrow x \sim a \).

\[
\begin{align*}
\bar{x} & = \bar{a} \\
x & \in \bar{x} \\
x & \in \bar{a} \\
\text{Therefore, } x & \sim a
\end{align*}
\]

Now show that \( x \sim a \rightarrow \bar{x} = \bar{a} \)

\[
\begin{align*}
x & \sim a \\
\text{Let } y & \in \bar{x} \\
y & \sim x \\
x & \sim a \\
y & \sim a \\
y & \in \bar{a} \\
\text{Therefore } \bar{a} & \subseteq \bar{x} \\
\text{Therefore } \bar{x} & = \bar{a}
\end{align*}
\]

(b) Proposition 2.4.4: Let \( \sim \) be an equivalence relation on set \( A \). Let \( a, b \in A \). Then sets \( \bar{a} \) and \( \bar{b} \) are either equal or are disjoint.

Proof (fill in the reasons):
If sets \( \bar{a} \) and \( \bar{b} \) are equal the proof is complete. So assume they are not equal.

Now assume sets \( \bar{a} \) and \( \bar{b} \) are not disjoint i.e. \( \bar{a} \cap \bar{b} \neq \emptyset \)

\[
\begin{align*}
\text{Let } x & \in \bar{a} \cap \bar{b} \\
x & \in \bar{a} \\
x & \sim a \\
\bar{x} & = \bar{a} \\
x & \in \bar{b} \\
x & \sim b \\
\bar{x} & = \bar{b} \\
\text{Therefore } \bar{a} & = \bar{b}
\end{align*}
\]

Contradiction. Therefore, they are either equal or disjoint.
6. Define $a \sim b$ if and only if $a^2 - b^2$ is evenly divisible by 3, for $a, b$ in the set of Integers $\mathbb{Z}$.

(a) Show that $\sim$ is an equivalence relation on $\mathbb{Z}$.

Work:

$R$: $a \sim a$ since $a^2 - a^2 = 0 = 3k \quad k \in \mathbb{Z}$

$S$: if $a \sim b$ then $a^2 - b^2 = 3k$

$\quad \Rightarrow b^2 - a^2 = 3(-k) \quad k \in \mathbb{Z}$

$T$: if $a \sim b$ and $b \sim c$ then $a^2 - b^2 = 3k$

$\quad b^2 - c^2 = 3l$

$\quad \frac{a^2 - c^2}{a^2 - c^2} = \frac{3k}{3l} = 3(k+l) \quad k+l \in \mathbb{Z}$

(b) Show the equivalence classes created by $\sim$.

Work:

$[0] = \{0, \pm 3, \pm 6, \pm 9, \ldots \} = 3\mathbb{Z}$

$[1] = \{\ldots, -5, -4, -2, -1, 1, 2, 4, 5, \ldots \} = 3\mathbb{Z} + 1$ or $3\mathbb{Z} + 2$

(c) Show the quotient set $\mathbb{Z}$ mod $\sim$.

$\mathbb{Z}/\sim = \{[0], [1]\}$
7. Given the set $A = \{4, 6, 8, 12, 18, 24\}$. For $a, b$ define \( \leq \) as: \( a \leq b \) if $a$ evenly divides $b$.

(a) Does this \( \leq \) define a partial order on $A$? Explain.

Work:

\[
\begin{array}{cccccc}
4 & 6 & 8 & 12 & 18 & 24 \\
\times & \times & \times & \times & \times & \times \\
6 & \times & \times & \times & \times & \times \\
8 & \times & \times & \times & \times & \times \\
12 & \times & \times & \times & \times & \times \\
18 & \times & \times & \times & \times & \times \\
24 & \times & \times & \times & \times & \times \\
\end{array}
\]

Yes

\[ a \leq a \] because $\frac{a}{a} = 1 \\
\text{if } \frac{a}{b} = k \in \mathbb{Z} \implies a = bk \\
\frac{b}{a} = \ell \in \mathbb{Z} \implies b = \ell a \\
\frac{c}{a} = \frac{k}{\ell} \implies a = \ell k \\
\implies a = b \]

(b) Draw the Hasse Diagram for $(A, \leq)$

Work:

\[ \begin{array}{c}
\text{24} \\
\text{18} \\
\text{12} \\
\text{6} \\
\end{array} \]