Notes on the Harmonic Oscillator and the Fourier Transform
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In these notes we first derive two well-known results and relate them in an elegant way. We show that the Hermite functions, the eigenfunctions of the harmonic oscillator, are an orthonormal basis for \( L^2 \), the space of square-integrable functions. Secondly we establish the Fourier inversion theorem on \( L^2 \). We then infer some simple properties of the Schwartz space of well-behaved functions.

I The Fourier Transform

Define the Fourier transform operator \( \mathfrak{F} \) as the linear transformation on integrable functions \( f \),
\[
(\mathfrak{F} f)(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-ipy} \, dy = \hat{f}(p) .
\] (I.1)
One sometimes writes \((\mathfrak{F} f)(p) = \hat{f}(p)\). The Fourier inversion theorem says \( \mathfrak{F}^2 f \) exists, and
\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mathfrak{F} f)(p) e^{ipx} \, dp = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(p) e^{ipx} \, dp .
\] (I.2)
The pair of identities (I.1) and (I.2) are the fundamental identities for Fourier transforms.

We show in these notes that these relations hold and have a meaning for arbitrary \( f \in L^2 \). The \( L^2 \)-inner product is
\[
\langle f, g \rangle_{L^2} = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx .
\] (I.3)

Square-integrable functions \( f \in L^2 \) are those of length \( \|f\|_{L^2} = \langle f, f \rangle_{L^2}^{1/2} < \infty \). A subset \( \mathcal{D} \subset L^2 \) is said to be dense, if any \( f \in L^2 \) can be approximated by a sequence \( f_n \in \mathcal{D} \). This means \( \lim_n \|f - f_n\|_{L^2} = 0 \). An orthonormal basis \( \{\Omega_n\} \) is a set of orthonormal vectors whose finite linear combinations are dense. A linear transformation \( T \) is continuous on \( L^2 \) if \( \|Tf\|_{L^2} \leq M \|f\|_{L^2} \) for some constant \( M < \infty \). A continuous transformation defined on a basis extends uniquely to all \( L^2 \).

II The Fourier Transform is Unitary

The transformation \( \mathfrak{F} \) is unitary on \( L^2 \). What does this mean? Define the matrix elements of the adjoint \( \mathfrak{F}^* \) of \( \mathfrak{F} \) by \( \langle f, \mathfrak{F}^* g \rangle_{L^2} = \langle \mathfrak{F} f, g \rangle_{L^2} \). For integrable functions \( f, g \), one can compute \( \mathfrak{F}^* \) as
\[
\langle f, \mathfrak{F}^* g \rangle_{L^2} = \langle \mathfrak{F} f, g \rangle_{L^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixy} f(y) \overline{g(x)} \, dxdy = \langle f, Pf \rangle_{L^2} .
\] (II.1)
Here we introduce the operator of reflection through the origin, namely
\[
(Pf)(x) = f(-x) .
\] (II.2)
Note that \( P\mathfrak{F} = \mathfrak{F} P \). The computation (II.1) shows that on integrable functions,
\[
\mathfrak{F}^* = P\mathfrak{F} = \mathfrak{F} P .
\] (II.3)
From these relations it is not clear that \( \mathfrak{F} \) has an inverse. We claim the striking fact that the inverse \( \mathfrak{F}^{-1} \) does exist, and moreover that it equals \( \mathfrak{F}^* \). In other words we claim that
\[
I = \mathfrak{F}^* \mathfrak{F} = \mathfrak{F} \mathfrak{F}^* .
\] (II.4)
The relations (II.4) state that \( \mathfrak{F} \) is unitary. We prove (II.4) in §VII using properties of the harmonic oscillator that we establish in §IV–§V.
III Some Consequences of Unitarity

The property that $\mathcal{F}$ is unitary has numerous beautiful consequences. We now mention two. The relation $I = \mathcal{F}^* \mathcal{F}$ in (II.4) is known as Plancherel’s theorem. Written in terms of expectations,

$$
\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(p)|^2 \, dp .
$$

(AIII.1)

A second consequence of the unitarity is the Fourier inversion formula. From (II.3–II.4) we infer

$$
I = \mathcal{F} \mathcal{P} \mathcal{F} ,
$$

(AIII.2)

or $f = \mathcal{F} \mathcal{P} \mathcal{F} f$, applied to any $f \in L^2$. Use the definitions to write out (AIII.2); it is just the desired inversion (I.2).

IV The Harmonic Oscillator

The operator

$$
a = \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right) , \quad \text{and its adjoint} \quad a^* = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right) ,
$$

(AIV.1)

are the lowering and raising operators for the harmonic oscillator with unit mass and frequency equal, and units with $\hbar = 1$. They satisfy $[a, a^*] = I$, and $[a, a^{*n}] = na^{*n-1}$. The operator

$$
H = a^* a = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 - 1 \right)
$$

(AIV.2)

is the corresponding Hamiltonian. Note that

$$
[H, a^{*n}] = a^{*n+1} a + a^* [a, a^{*n}] - a^{*n} H = na^{*n} .
$$

(AIV.3)

The differential equation $\sqrt{2} a \Omega_0(x) = \Omega_0(x) + \Omega_0'(x) = 0$ has the normalized solution

$$
\Omega_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2} ,
$$

(AIV.4)

and yields a zero-energy ground state of $H$ satisfying $H \Omega_0 = a^* a \Omega_0 = 0$.

Introduce the Hermite functions $\Omega_n(x)$ and the Hermite polynomials $H_n(x)$ defined by

$$
\Omega_n(x) = \frac{1}{\sqrt{n!}} a^{*n} \Omega_0(x) = \frac{1}{2^{n/2} \sqrt{n!}} H_n(x) \Omega_0(x) .
$$

(AIV.5)

From (AIV.3), we infer that $\Omega_n$ is an eigenfunction of $H$ with eigenvalue $n$. Furthermore

$$
\langle \Omega_n, \Omega_n \rangle = \frac{1}{n} \langle \Omega_{n-1}, aa^* \Omega_{n-1} \rangle = \frac{1}{n} \langle \Omega_{n-1}, (H + I) \Omega_{n-1} \rangle = \langle \Omega_{n-1}, \Omega_{n-1} \rangle = \cdots = \langle \Omega_0, \Omega_0 \rangle = 1 ,
$$

(AIV.6)

so using the fact that the eigenvalues of the $\Omega_n$ are different for different $n$, these functions satisfy

$$
\langle \Omega_n, \Omega_m \rangle = \delta_{nm} .
$$

(AIV.7)

In other words, the Hermite functions are an orthonormal set of eigenfunctions of $H$, with the eigenvalues are $n = 0, 1, 2, \ldots$.
V The Oscillator Eigenfunctions are a Basis

If the \{\Omega_n\} are not a basis, then there is a non-zero vector \( \chi \in L^2 \) perpendicular to all of the \( \Omega_n \). We assume that there exists a vector \( \chi \in L^2 \) satisfying \( \chi \perp \Omega_n \) for all \( n \). We then show that \( \chi = 0 \).

We use the generating function for Hermite polynomials, namely \( G(z; x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) \). We now compute \( G(z; x) \), which also was a homework problem. Use the definition (IV.5) to write \( G(z; x) = \Omega_0^{-1} e^{\sqrt{2z}a^*} \Omega_0 \). For any constant \( \lambda \), the “translation identity” \( e^{-\lambda a^*} e^\lambda a = a^* - \lambda \) follows from expanding each side of this equality as a power series in \( \lambda \). Now define \( f(\lambda) = e^{\lambda a^*} e^{\lambda a} e^{-\lambda(a^* + a)} e^{\lambda^2/2} \). Using the translation identity, we infer \( f(\lambda) = f(0) = 1 \), and we have proved the “rearrangement identity” \( e^{\lambda a^*} e^{\lambda a} = e^{\lambda(a^* + a)} e^{-\lambda^2/2} \). Since also \( e^{\sqrt{2z}a} \Omega_0 = \Omega_0 \), we apply the rearrangement identity with \( \lambda = \sqrt{2z} \) to show \( e^{\sqrt{2z}a^*} \Omega_0 = e^{-z^2 - 2zx} \Omega_0 \).

Therefore dividing by \( \Omega_0 \),

\[
G(z; x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) = e^{-z^2 + 2zx} \tag{V.1}
\]

Suppose we are given a vector \( \chi \) orthogonal to all the \( \Omega_n \). Using (V.1), we have for any real \( p \),

\[
\sum_{n=0}^{\infty} \frac{(-ip/\sqrt{2})^n}{\sqrt{n!}} \langle \Omega_n, \chi \rangle L^2 = \int_{-\infty}^{\infty} G(-ip/2; x) \Omega_0(x) \chi(x) dx = \sqrt{2\pi} e^{ip^2/4} \mathcal{F} \chi(\Omega_0) (p) = 0 \tag{V.2}
\]

Hence \( \mathcal{F} \chi(\Omega_0) = 0 \). Therefore, unless \( \chi = 0 \), the Fourier transform \( \mathcal{F} \) has no inverse!

In order to show that \( \chi = 0 \), multiply the (vanishing) Fourier transform of \( \chi \Omega_0 \) by \( e^{ipa} \). Then

\[
\int_{-\infty}^{\infty} \chi(x) \Omega_0(x) e^{-ip(x-a)} dx = 0 \tag{V.3}
\]

Take \( \epsilon > 0 \). Then multiply (V.3) by \( e^{-a^2} \), and integrate over \( p \). By completing the square for the Gaussian \( p \)-integration, we obtain

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi(x) \Omega_0(x) e^{-\epsilon p^2 - ip(x-a)} dx dp = \frac{\sqrt{\pi}}{\epsilon} \int_{-\infty}^{\infty} \chi(x) \Omega_0(x) e^{-(x-a)^2/4\epsilon} dx = 0 \tag{V.4}
\]

Let \( \epsilon \to 0 \), and recall that \( \sqrt{\frac{1}{\epsilon \pi}} e^{-(x-a)^2/\epsilon} \to \delta(x-a) \). Thus \( \chi(a) \Omega_0(a) = 0 \), for all \( a \). But \( \Omega_0(a) > 0 \), so divide by this function to conclude \( \chi(a) = 0 \). This is a contradiction to the assumption that \( \chi \neq 0 \). Therefore we conclude that the \( \{\Omega_n\} \) form a basis.

VI Oscillator Eigenfunctions are Fourier Eigenfunctions

Recall that the operators in (IV.1) denote the lowering and raising operators for the simple harmonic oscillator. Note \( Pa^* = -a^*P \). Thus using (I.1), (II.3), and \( P^2 = I \), we obtain the commutation relations

\[
\mathfrak{F} a = -ia \mathfrak{F} , \quad \text{and} \quad \mathfrak{F} a^* = ia^* \mathfrak{F} \tag{VI.1}
\]

As a consequence, the oscillator Hamiltonian \( H = a^*a = \frac{1}{2} (-d^2/dx^2 + x^2 - 1) \) commutes with \( \mathfrak{F} \), and \( \mathfrak{F} \) and \( H \) can be simultaneously diagonalized. In other words, the oscillator eigenfunctions \( \Omega_n \) of \( H \) are also eigenfunctions of \( \mathfrak{F} \). The corresponding eigenvalues are \( \pm 1, \pm i \), as follows from the representation of the \( n \)th-eigenfunction \( \Omega_n = \frac{1}{\sqrt{n!}} a^n \Omega_0 \). Using (VI.1) and \( \mathfrak{F} \Omega_0 = \Omega_0 \), we infer that

\[
\mathfrak{F} \Omega_n = \frac{1}{\sqrt{n!}} \mathfrak{F} a^n \Omega_0 = i^n \frac{1}{\sqrt{n!}} a^n \mathfrak{F} \Omega_0 = i^n \Omega_n \tag{VI.2}
\]
VII  The Fourier Inversion Theorem

From (VI.2) and 
\[ P \Omega_n = (-1)^n \Omega_n, \]  
we conclude that 
\[ \mathcal{F} P \mathcal{F} \Omega_n = \Omega_n. \]  
Thus  
\[ \mathcal{F} P \mathcal{F} = I, \]  
on the span of the functions \( \Omega_n \). (VII.1)

Using the discussion in §I and the continuity of the identity \( I \), we infer that the equality (VII.1) extends to all \( L^2 \) if and only if the collection of harmonic oscillator eigenfunctions \( \{ \Omega_n \} \) are a basis for \( L^2 \). In that case, (VII.1) shows that \( \mathcal{F} \) has a left and right inverse \( \mathcal{F} P = P \mathcal{F} = \mathcal{F}^* \). But in §V we established that the \( \{ \Omega_n \} \) are a basis. So we conclude that (VII.1) extends of \( L^2 \) and \( \mathcal{F}^{-1} = \mathcal{F}^* \).

VIII  Schwartz Space of Well-Behaved Functions

The Schwartz space \( \mathcal{S} \) of well-behaved functions is defined as those functions \( f(x) \in L^2 \) for which \( H^n f \in L^2 \) for all \( n = 0, 1, 2, 3, \ldots \), where \( H \) is the oscillator Hamiltonian. Since we showed in §VI that \([H, \mathcal{F}] = 0\), we conclude that \( \mathcal{F} \mathcal{S} \subset \mathcal{S} \). But as \( \mathcal{F}^{-1} = \mathcal{F} P \), and \( P \mathcal{S} \subset \mathcal{S} \), it follows that \( \mathcal{F} \mathcal{S} = \mathcal{S} \). (VIII.1)

Note that \( \Omega_n \in \mathcal{S} \), for every \( n \). Furthermore any \( f \in L^2 \) has an expansion in the \( \{ \Omega_n \} \)-basis, 
\[ f(x) = \sum_{j=0}^{\infty} f_j \Omega_j(x). \]  
(VIII.2)

We write the condition that \( f \in L^2 \) actually is an element of \( \mathcal{S} \) as  
\[ \|H^n f\|_{L^2}^2 = \sum_{j=0}^{\infty} j^{2n} |f_j|^2 < \infty, \]  
(VIII.3)

for all \( n = 0, 1, 2, 3, \ldots \). In other words, the Schwartz space has “rapidly decreasing” coefficients in the oscillator/Fourier basis \( \{ \Omega_n \} \).

Any \( f \in L^2 \) has a sequence of approximating functions \( f_R \in \mathcal{S} \). In fact for \( f \) of the form (VIII.2), define  
\[ f_R(x) = \sum_{j=0}^{\infty} f_j e^{-j^2/R} \Omega_j(x). \]  
(VIII.4)

Clearly \( f_R \in \mathcal{S} \). Furthermore, as \( e^{-j^2/R} \) is monotonically decreasing in \( j \),  
\[ \|f_R - f\|_{L^2}^2 = \sum_{j=1}^{\infty} |f_j|^2 \left( 1 - e^{-j^2/R} \right)^2 \leq \|f\|_{L^2}^2 \left( 1 - e^{-1/R} \right)^2 \to 0, \]  
as \( R \to \infty \). (VIII.5)

Exercise. Show that the Schwartz functions are exactly those for which  
\[ \sup_{x \in \mathbb{R}} \left| x^r \frac{d^s}{dx^s} f(x) \right| < \infty, \]  
(VIII.6)

for all \( r, s = 0, 1, 2, 3, \ldots \).