Name: Solutions!

Math 315: Advanced Calculus
Solutions #4

1. Section 5.1 #3(a),(d)

#3 (a) The function is not defined at $x = 0$, so it is not differentiable there.

(d) It is differentiable with derivative 0.

**Proof:** Certainly $x = 0$ is in the domain. To see if $f(x)$ is differentiable at $x = 0$, we check the limit as $x \to 0$ of the appropriate ratio:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

since $f(0) = 0^2 = 0$. We claim that this limit is 0. Choose $\epsilon > 0$. We claim there exists $\delta > 0$ such that $|\frac{f(x)}{x} - 0| < \epsilon$ for all $|x| < \delta$. Let $\delta = \epsilon$. Then

$$|\frac{f(x)}{x}| = \begin{cases} \frac{x^2}{x} = x & \text{if } x \text{ is rational, and} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

In either case, if $|x| < \delta = \epsilon$, then $|\frac{f(x)}{x}| < \epsilon$. 

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2. Section 5.1 #7

If \( f \) is differentiable at \( x = a \) with \( f'(a) = b \), then

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = b
\]

Subtract \( b \) from both sides of the equation and combine terms. Notice that \( \lim_{x \to a} b = b \), so we can bring \( b \) inside the limit sign on the left.

On the other hand, suppose

\[
\lim_{x \to a} \frac{f(x) - f(a) - b(x - a)}{x - a} = 0.
\]

The left hand side equals

\[
\lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} - b \frac{x - a}{x - a} \right) = \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} - b \right).
\]

Careful, you cannot evaluate this limit term-wise \textit{a priori} because the limit of the sums is the sum of the limits only when both limit are defined! However, we know the limit exists (and is 0) which allows us to do the following trick:

\[
\lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} - b \right) + b = 0 + b
\]

\[
= \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} - b + b \right)_{\text{since } \lim_{x \to a} b = b}
\]

\[
= \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right) = b,
\]

and we have proven what we wanted.
3. Section 5.1 # 10(b),(g)

(b) Rewrite the expression as

\[ \lim_{h \to 0} \frac{2f(x + h) - 2f(x - h) + f(x + h) - f(x)}{5h} \]

and use Example 5.1.11 to prove that this equals \( f'(x) \).

This limit might exist without \( f \) being differentiable: consider

\[ f(x) = \begin{cases} 2x & x \geq 0 \\ -3x & x < 0 \end{cases} \]

One can check that the given limit is defined at \( x = 0 \), but \( f(x) \) is not differentiable at \( x = 0 \).

(g) Rewrite the expression as

\[ \lim_{h \to 0} \frac{f(x - ah) - f(x)}{(b - a)h} - \frac{f(x - bh) - f(x)}{(b - a)h} \]

which equals (you can check this!)

\[ \lim_{h \to 0} \frac{-a}{b - a} \frac{f(x - ah) - f(x)}{(-ah)} - \frac{b}{b - a} \frac{f(x - bh) - f(x)}{bh} \]

which simplifies to \( \frac{-a}{b - a} f'(x) + \frac{b}{b - a} f'(x) = f'(x) \).

This limit might exist without \( f \) being differentiable: Suppose \( a < 0 \) and \( b > 0 \). consider

\[ f(x) = \begin{cases} bx & x \geq 0 \\ -ax & x < 0 \end{cases} \]

One can check that the given limit is defined at \( x = 0 \), but \( f(x) \) is not differentiable at \( x = 0 \).

On the other hand, suppose \( a \) and \( b \) have the same sign but are not equal. Then \( f \) must be differentiable for this limit to exist.
4. Section 5.2 #1(i),(l)

(i) \( f'(x) = 2 - \sin x \). Then \( g'(x) = f''(x) = -\cos x \). Then \( g'(1) = -\cos 1 \).

(l) If \( f(x) \) is even, then \( f(-x) = f(x) \). Then

\[
 f'(-x) = \lim_{h \to 0} \frac{f(-x + h) - f(-x)}{h} \\
= \lim_{h \to 0} \frac{f(x - h) - f(x)}{h} \\
= -\lim_{h \to 0} \frac{f(x - h) - f(x)}{-h} = -f'(x).
\]

So \( f' \) is odd. The proof is similar for the other one.
5. Section 5.2 #10(a),(b)

(a) First part of the solution is thanks to Darryll Creel:
Suppose $f'(x) \geq 0$ on $(a, b)$. We show $f$ is increasing. Since $f$ is differentiable on the interval, it is also continuous. Choose $x_1, x_2 \in (a, b)$ any points such that $a < x_1 < x_2 < b$. By the mean value theorem, there exists some $x \in (x_1, x_2)$ such that

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \quad (1)$$

Since $f'(x) \geq 0$ and the denominator is greater than 0, we have $f(x_2) \geq f(x_1)$. Since this relation holds for any $x_2 > x_1$ in the interval, $f$ is increasing.

You cannot reason backwards using the MVT! In particular, you can’t conclude that, if $f$ is increasing, the numerator and the denominator of Equation (1) are positive, which implies $f'(x) \geq 0$. The reason is that this doesn’t hold for every $x$ in the interval.

The other direction is due to Lars Aiken. Assume $f$ is differentiable and increasing on $(a, b)$. Then for any $c \in (a, b)$,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}.$$

As $x \to c^+$, we know the numerator is nonnegative since $f$ is increasing, and the denominator is positive since $x$ approaches $c$ from the right. Thus $f'(c) \geq 0$. Since we did this for any $c$, it follows that $f'(x) \geq 0$.

(b) We assume that $f$ is differentiable. $f'(x) > 0$ is not equivalent to $f$ being strictly increasing on $(a, b)$. $f$ may be strictly increasing and yet the derivative equal 0 at some point. Consider $f(x) = x^3$ and look at its behavior near $x = 0$. 


6. Section 5.2 #13(a) \( f' \) bounded on \((a, b)\) implies that there exists \( M \) such that
\[
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \leq M.
\]
We would like to say that this means that \( \frac{f(x) - f(c)}{x - c} \) is bounded for any \( x, c \in (a, b) \), conclude that \( f(x) \) is Lipschitzian and therefore uniformly continuous. But this ratio is not bounded just because the limit is bounded!

Instead, we use the Mean Value Theorem. Let \( x, t \in (a, b) \) with \( x \neq t \) and without loss of generality assume \( x < t \). Then by the MVT, there exists \( c \in (x, t) \) such that
\[
f'(c) = \frac{f(x) - f(t)}{x - t}
\]
But \( f' \) is bounded on \((a, b)\) implies that this whole expression is less than \( M \) for all \( x, t \in (a, b) \). Thus \( f \) is Lipschitzian and therefore uniformly continuous.

7. Section 5.2 #20(f) We use the Mean Value Theorem. Let \( f(x) = \ln(x + 1) \) on the interval \( x \geq 0 \). Then \( f'(x) = \frac{1}{x+1} \). By the MVT, there exists some \( c \in (0, x) \) such that
\[
f'(c) = \frac{f(x) - f(0)}{x} = \frac{\ln(x + 1) - \ln 1}{x} = \frac{\ln(x + 1)}{x},
\]
or \( x \frac{1}{c+1} = \ln(x + 1) \). But then \( 0 < c < x \) implies \( x \frac{1}{x+1} \leq x \frac{1}{c+1} \leq x \frac{1}{1} \) which gives the desired inequality. When \( x = 0 \), the inequality is an equality, as all terms are 0. When \( x \neq 0 \), \( \frac{x}{x+1} < \frac{x}{c+1} \) as \( c < x \), and so \( \frac{x}{x+1} < \ln(x + 1) \) and similarly \( \frac{x}{c+1} < x \) as \( c > 0 \), which implies \( \ln(x + 1) < x \) when \( x \neq 0 \).
8. Section 5.2 #37 Consider the function \( h(x) = e^{-x} - \sin x \). Suppose that \( r_1 \) and \( r_2 \) are consecutive roots of \( f(x) \). Then \( f(r_1) = 1 - e^{r_1} \sin r_1 = 0 \) implies \( e^{r_1} \sin r_1 = 1 \), or \( \sin r_1 = e^{-r_1} \). Then \( h(r_1) = e^{-r_1} - \sin r_1 = 0 \). Similarly \( h(r_2) = 0 \). Then by Rolle’s theorem, there exists some \( c \in (r_1, r_2) \) such that \( h'(c) = 0 \). But \( h'(x) = -e^{-x} - \cos x \). Thus \( h'(c) = -e^{-c} - \cos c = 0 \) implies \( \cos c = -e^{-c} \) or \( e^{c} \cos c = -1 \), which implies that \( g(c) = 1 + e^{c} \cos c = 0 \), and so \( g(x) \) has a root in between \( r_1 \) and \( r_2 \).

9. Section 5.3 #2(a)

To determine concavity and points of inflection, we have to find \( f'(x) \) and \( f''(x) \). For \( x \neq 0 \),

\[
f'(x) = \frac{2x(x^3) - 3x^2(x^2 - 1)}{(x^3)^2} = \frac{3 - x^2}{x^4}
\]

and then

\[
f''(x) = \frac{-2x(x^4) - 4x^3(3 - x^2)}{x^8} = \frac{2x^2 - 12}{x^5}.
\]

To find the derivative at 0, we have to directly calculate a limit:

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{x^2 - 1}{x^3x}
\]

if the limit is finite. However, the limit is \( \infty \) and so \( f'(0) \) does not exist. Therefore \( f''(x) \) doesn’t exist either.

Now we are prepared to check concavity. We are interested in when \( f''(x) > 0 \) and when \( f''(x) < 0 \). Points of inflection may occur when the sign changes. So we first ask when \( f''(x) = 0 \). This is when \( x^2 - 6 = 0 \), or \( x = \pm \sqrt{6} \). When \( x < -\sqrt{6} \), we note that \( f''(x) < 0 \) and so the curve is concave down. The next region we need to worry about is \(-\sqrt{6} < x < 0\). (We cannot jump to \(-\sqrt{6} < x < \sqrt{6}\) because \( f''(x) \) is not defined at \( x = 0 \) so funny stuff can happen there!). On \(-\sqrt{6} < x < 0\), \( f''(x) > 0 \) and on \( 0 < x < \sqrt{6}\), \( f''(x) < 0 \). Lastly, for \( x > \sqrt{6}\), \( f''(x) > 0 \). Therefore, \( f(x) \) is concave up when \( x \in (-\sqrt{6}, 0) \cup (\sqrt{6}, \infty) \) and \( f(x) \) is concave down when \( x \in (-\infty, -\sqrt{6}) \cup (0, \sqrt{6}) \).

To check the point of inflection, we have to be careful that the rule that \( f(x) \) changes concavity when \( f''(x) \) changes sign is only true at
points where \( f \) is \textit{continuous}. Since \( f \) is continuous except at \( x = 0 \), we conclude that there are points of inflection at \( x = -\sqrt{6} \) and \( x = \sqrt{6} \). We check for the possibility of a point of inflection at \( x = 0 \) by looking at the graph. Since \( \lim_{x \to 0^+} f(x) = -\infty \) and \( \lim_{x \to 0^-} f(x) = \infty \), there is no point of inflection at \( x = 0 \).

I don’t include the graph because it’s too hard to do on the computer.

10. Section 5.3 #11 We assume that \( f \) is uniformly differentiable, that is:

\[
\left| \frac{f(x) - f(t)}{x-t} - f'(x) \right| < \epsilon
\]

whenever \( |x - t| < \delta \). Keep in mind that this definition is symmetric in \( x \) and \( t \), as this will become handy. We want so show that \( f' \) is continuous, i.e. for all \( c \in (a, b) \) and all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that

\[
|f'(x) - f'(c)| < \epsilon \text{ whenever } |x - c| < \delta.
\]

We notice that

\[
|f'(x) - f'(c)| = |f'(x) - f(x) - f'(c)(x - c)|
\]

\[
\leq |f'(x) - f(x) - f(c)(x - c)| + |f(x) - f(c) - f'(c)(x - c)|
\]

and so we try to bound the two terms on the right. Choose \( \epsilon > 0 \). By the fact that \( f \) is uniformly differentiable, choose \( \delta_1 > 0 \) such that

\[
|f'(x) - \frac{f(x) - f(c)}{x-c}| < \epsilon/2 \text{ whenever } |x - c| < \delta_1
\]

and \( \delta_2 > 0 \) such that

\[
\left| \frac{f(x) - f(c)}{x-c} - f'(c) \right| < \epsilon/2 \text{ whenever } |x - c| < \delta_2.
\]

Now choose \( \delta = \min\{\delta_1, \delta_2\} \) and then for \( |x - c| < \delta \) we have \( |f'(x) - f'(c)| < \epsilon/2 + \epsilon/2 = \epsilon \) and we have proven the desired result. I should point out just for the sake of clarity that in fact, you can choose \( \delta = \delta_1 = \delta_2 \), by the symmetry of uniform differentiability. However, since
this requires some reflection, I just chose the \( \delta \)'s separately so that it would be easier to see.

(b) The converse is not true – just because \( f' \) is continuous does not mean that \( f \) is uniformly differentiable. Consider \( f(x) = \frac{1}{x} \) on the interval (0,1). The derivative \( f'(x) = -\frac{1}{x^2} \) is clearly continuous on (0,1), but \( f \) is not uniformly differentiable on (0,1), as you can check (and should!)

11. Section 5.3 #12 Note that \( f'(x) = -\frac{1}{x^2} \). We are required to show that for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\left| \frac{1}{x} - \frac{1}{t} + \frac{1}{x^2} \right| < \varepsilon
\]

whenever \( 0 < |x - t| < \delta \), with \( x, t \in (a, \infty) \). We compute directly:

\[
\frac{1}{x} - \frac{1}{t} + \frac{1}{x^2} = \frac{t-x}{xt} + \frac{1}{x^2} = \frac{-1}{xt} + \frac{1}{x^2} = \frac{1}{x^2} \left( \frac{x}{t} + 1 \right) \leq \frac{1}{a^2} \left( 1 - \frac{x}{t} \right)
\]

where the last inequality comes from \( x > a > 0 \). Choose \( \varepsilon > 0 \). We need to find \( \delta \) such that \( \left| \frac{1}{a^2} (1 - \frac{x}{t}) \right| < \varepsilon \) when \( |x - t| < \delta \). We note that \( |x - t| < \delta \) implies \( -\delta + t < x < \delta + t \) and so \( |1 - \frac{x}{t}| \leq 1 - \frac{t + \delta}{t} = \frac{\delta}{t} \leq \frac{\delta}{a} \). Let \( \delta = \varepsilon a^3 \). Then \( |x - t| < \delta \) implies

\[
\left| \frac{1}{a^2} (1 - \frac{x}{t}) \right| \leq \left| \frac{1}{a^3} \right| < \frac{\varepsilon a^3}{a^3} = \varepsilon.
\]

This part is due to Toni Robertson: On \( (0, \infty) \), the function is not uniformly differentiable. Construct sequences \( x_n \) and \( t_n \) such that \( |x_n - t_n| < 1/n \) but there exists \( \varepsilon > 0 \) such that \( \left| \frac{1}{x_n} - \frac{1}{x_n t_n} \right| \geq \varepsilon \). Let \( x_n = \frac{1}{n} \) and \( t_n = \frac{1}{2n} \). Then \( |x_n - t_n| = \frac{1}{2n} < \frac{1}{n} \) for \( n > 1 \). However,

\[
\left| \frac{1}{x_n^2} - \frac{1}{x_n t_n} \right| = |n^2 - 2n^2| = |n^2|
\]

diverges.
12. Section 5.3 #18 Follow the hint in the back of the book and it just falls out!

13. Section 5.3 #29 Since \( f(x) \) has \( n+1 \) 0’s, it follows from Rolle’s theorem that there are \( n \) places \( y_1, \ldots, y_n, y_i \in (x_{i-1}, x_i) \) where \( f'(x) \) is 0. Then \( f'(x) \) satisfies the conditions of Rolle’s theorem by hypothesis, and thus Rolle’s theorem implies \( f''(x) \) has \( n-1 \) 0’s at \( z_1, \ldots, z_{n-1} \), where \( z_i \in (y_i, y_{i+1}) \). Continue this way to find that \( f^{(k)} \) has \( n-k+1 \) 0’s, and finally that \( f^{(n)} \) has at least one 0.