Parametric Equations, Function Composition
and the Chain Rule: A Worksheet

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1 Parametric Equations

We have seen that the graph of a function $f(x)$ of one variable consists of a set of points in the $xy$-plane. These points are the set

$$\{(x, f(x)) : x \in D\}$$

where $D$ is the domain of $f(x)$.

The graph of a function has many properties. For example, every such graph passes the “vertical line test”. This test reflects the fact that, for every $x$ value, there is exactly one $y$ value, mainly $f(x)$. We notice that $x$ is always the “input” and $y$ is the “output” that we get by evaluating $f(x)$.

Example 1.1 *How is the* graph of a function *different from a function?*

Solution: The function describes a way to get a number out for each $x$ value you put in. The function doesn’t “live” in the plane. The graph of a function, on the other hand, describes the set of points $\{(x, f(x))\}$ in the $xy$-plane.

Parametric equations are just another way of describing a set of points in the $xy$-plane in our case (or in higher dimensions in general). Instead of describing these points by $\{(x, f(x))\}$, we describe the points by the set $\{(x, y)\}$, where $x$ itself (as well as $y$) is determined by a function $x(t)$ (or $y(t)$). Here $t$ is the “input”, and it is called a “parameter”. This is similar to how $x$ is the input in the
case of a graph of a function. Similarly, $y$ is actually a function $y(t)$ dependent on the parameter $t$. Given a particular value of $t$, one can find a point in the $xy$-plane by evaluating $(x(t), y(t))$. Another way that people write parametric equations is

$$x = f(t) \text{ and } y = g(t)$$

for some range of $t$ values. The functions $f(t)$ and $g(t)$ replace the functions $x(t)$ and $y(t)$, respectively, but they are just different names for the same functions.

Example 1.2 *Parametric Equations (Basic)*

One use of parametric equations is that it doesn’t rely on the resulting points $\{(x, y)\}$ to actually be a graph of a function. For example, the parametric equations

$$x = 1 \text{ and } y = t, \ t \in [0, 4]$$

describes a vertical line segment given by points $\{1, y\}$ where $y$ goes from 0 to 4. This obviously doesn’t pass the vertical line test and could not be the graph of a function.

Example 1.3 You Try It

1. Describe the vertical line segment that goes from $(3, 7)$ to $(3, 14)$ using parametric equations.

2. Do problem 21 p. 65

Another important example is the case of a circle of radius $r$.

Example 1.4 You Try It

Find parametric equations for the circle of radius 5 centered around the origin. If you have trouble, consult the book on page 61.

You can find the formal definition of parametric equations in your text. The “equations” are $x = f(t)$ and $y = g(t)$. They are called “parametric” because they depend on a parameter $t$. 

2
Remark 1 Any particular point \((x, y)\) on the curve described by parametric equations
\[
x = f(t) \quad \text{and} \quad y = g(t)
\]
is obtained by a particular choice of \(t\). You cannot use one choice of \(t\) for finding \(x\) and a different choice for finding \(y\). This is illustrated in the next example.

Example 1.5 The motion of a fly is described by the equations
\[
\begin{align*}
x &= -\cos(t) \\
y &= \sin(t), \quad t \in [0, 2\pi]
\end{align*}
\]
At what time is the fly at the position \(\left(\sqrt{2} \over 2, \sqrt{2} \over 2\right)\)?

Solution: “At what time” means we’re looking for a value of \(t\) that gives us the point \((x, y) = \left(\sqrt{2} \over 2, \sqrt{2} \over 2\right)\). This expression
\[
(x, y) = \left(\sqrt{2} \over 2, \sqrt{2} \over 2\right)
\]
is really two equations that we need to solve using the fact that \(x = \cos(t)\) and \(y = \sin(t)\), mainly
\[
-\cos(t) = \sqrt{2} \over 2
\]
and
\[
\sin(t) = \sqrt{2} \over 2.
\]
However we need to solve these equations simultaneously, i.e. the fly has to be in the appropriate \(x\) position and in the appropriate \(y\) position at the same time.

In the first equation, we find that \(t = 3\pi \over 4\) and \(t = 5\pi \over 4\) are two solutions to the equation. For the second equation, we find that \(t = \pi \over 4\) and \(t = 3\pi \over 4\) are both solutions. The only simultaneous solution is \(t = 3\pi \over 4\), when both equations are satisfied for the same \(t\) value.
2 Composition of Functions and Parametric Equations

Recall that if \( f(x) \) and \( g(x) \) are two functions and the range of \( g(x) \) is in the domain of \( f(x) \), then we can form the composition

\[
f(g(x)).
\]

The goal of this section is to understand this composition of functions better.

Example 2.1 This example illustrates the calculations involved with composition.

Find the composition \( f(g(x)) \) when \( f(x) = \sin x^3 \) and \( g(x) = \frac{1}{x^2} \).

Solution: First make sure you’re clear on the notation: \( \sin x^3 = \sin(x^3) \) which is NOT the same as \( \sin^3 x = (\sin x)^3 \).

\[
f(g(x)) = \sin (g(x)^3) = \sin \left[ \left( \frac{1}{x^2} \right)^3 \right] = \sin \left( \frac{1}{x^6} \right)
\]

or alternatively

\[
f(g(x)) = f \left( \frac{1}{x^2} \right) = \sin \left[ \left( \frac{1}{x^2} \right)^3 \right] = \sin \left( \frac{1}{x^6} \right).
\]

Example 2.2 You Try It

1. Do Problems 37, 39 on p. 22.

Example 2.3 The profit made on orange juice as a function of volume. This example illustrates the concept of composition.

Imagine that \( x \) is an amount (volume) of orange juice in litres. Let \( g(x) \) be the price of buying \( x \) litres of orange juice. Suppose that \( g(x) = 3x \), so it costs $3 per litre of orange juice. Now suppose that \( f(x) \) is the amount of money the company earns when collecting \$x. For example, \( f(x) = .2x \), i.e. the company has a profit of 20 cents for each dollar collected. Notice that \( x \) does NOT stand for the same thing in the context of \( f(x) \) and in \( g(x) \). As a “variable” in
the domain of $g(x)$, $x$ is an amount of orange juice. As a variable in the domain of $f(x)$, $x$ is a quantity of money. The important thing, however, is that the range of $g(x)$ is the domain of $f(x)$; both are measured in dollars. Now what does $f(g(x))$ mean? Since $x$ is first taken in by the function $g(x)$ (on the inside), we know that $x$ must stand for an amount of orange juice. Now $g(x)$ is the price of that orange juice, and $f(g(x))$ is the amount of profit taken in for that price. Thus $f(g(x))$ represents the amount of profit obtained when $x$ litres of orange juice are sold.

Here’s the explicit calculation:

$$f(g(x)) = 0.2g(x) = 0.2(3x) = 1.5x$$

or alternatively,

$$f(g(x)) = f(3x) = 0.2(3x) = 1.5x.$$

Now let’s do a parametric composition.

Example 2.4 Suppose that an ant moves along the graph of the parametric equations given by

$$x = 2\cos t \text{ and } y = 3\sin t \text{ where } t \in [0, 2\pi).$$

at any time $t$ in the domain. First, convince yourself that this is an ellipse by finding the Cartesian equation that these parametric equations satisfy. Notice that $3x = 6\cos t$ and $2y = 6\sin t$, which is like a circle of radius 6. From this, you might guess that $(3x)^2 + (2y)^2 = 36$. Dividing both sides by 36, you’ll find the equation of an ellipse in standard form.

Now suppose that the position of a bird depends on the position of the ant. Suppose that the bird can be found at the position $x_{\text{bird}} = 2x_{\text{ant}}$ and $y_{\text{bird}} = 5y_{\text{ant}}$. Can you figure out the position of the bird as a function of time?

Since the bird’s position depends on the ant’s position, which in turn depends on time, one suspects this is a composition question, as compositions always reflect a “chain” of dependencies. In this case, we see that $x_{\text{bird}} = 2x_{\text{ant}} = 2(2\cos t) = 4\cos t$. Similarly, $y_{\text{bird}} = 2y_{\text{ant}} = 2(3\sin t) = 6\sin t$. Thus the bird is also on a (different) ellipse in the $xy$-plane.
Here is a picture of what’s going on when finding $f(g(x))$. Take $x$ and first apply $g$ to it, to get $g(x)$:

$$x \rightarrow g(x)$$

Now apply $f$ to the value $g(x)$, to get $f(g(x))$:

$$g(x) \rightarrow f(g(x)).$$

Example 2.5 Recognizing functions as compositions. How do you recognize a function as a composition of others? There may be more than one answer!!

1. If you’re given $f(x) = x^7 \sin x^2$, this is NOT the composition of $x^7$ and $\sin x^2$, but the product of these two functions. However, here is one way of seeing a composition in here: $\sin x^2$ is a composition. Let $g(x) = x^2$, and $h(x) = \sin x$. Then $h(g(x)) = \sin g(x) = \sin x^2$. Notice that $g(h(x)) = (h(x))^2 = (\sin x)^2 = \sin^2 x$, which is not the same as $\sin x^2$, so the order of composition is important.

2. Again, let $f(x) = x^7 \sin x^2$. Let $g(x) = x^2$ and $h(x) = x^7 \sin x$. Then

$$h(g(x)) = (g(x))^7 \sin g(x) = (x^2)^7 \sin x^2 = x^7 \sin x^2 = f(x).$$

so indeed we can see $f(x)$ as a composition. However, this is not very useful from the point of view of differentiation, since $h(x)$ is hard to differentiate.

3. If you’re given $f(x) = (2x + 1)^2$, then the function on the “inside” is $g(x) = 2x + 1$. What is being “done” to $g(x)$? It’s being squared. What function takes anything and squares it? $h(x) = x^2$. Thus $h(g(x)) = (2x + 1)^2 = f(x)$.

4. $f(t) = (1 + \cos 2t)^{-4}$. The “inside” is $g(t) = 1 + \cos 2t$. The outside function takes whatever is on the inside to the -4th power. Thus $h(x) = x^{-4}$. Then

$$h(g(t)) = (g(t))^{-4} = (2 \cos 2t)^{-4}.$$
Notice that it didn’t matter that we wrote $h(x)$ as a function of $x$. Since $x$ is just a variable, it stands for “whatever you feed into $h$”.

Notice also in this case that $g(t)$ is also a composition of functions. If we let $k(t) = 2t$ and $m(t) = 1 + \cos t$, then $g(t) = m(k(t)) = 1 + \cos(k(t)) = 1 + \cos 2t$.

5. $f(t) = (1 + \cos 2t)^{-4}$ (again). Notice that you could have broken the function up differently to begin with. Let $g(t) = 2t$. Let $h(x) = (1 + \cos x)^{-4}$. Then

$$h(g(t)) = h(2t) = (1 + \cos 2t)^{-4} = f(t).$$

This way of breaking up the equation is less useful from the point of view of differentiating functions, since $h(x)$ is hard to differentiate without using the chain rule again. We’ll see more on the chain rule below.

Example 2.6 You Try It

1. Do Exercise 41 on p. 22

3 Taking the Derivative of the Composition of Functions: the Chain Rule

Theorem 3.1 (The Chain Rule) The chain rule specifies how to differentiate a composition of functions. Let $f(x) = h(g(x))$. Then

$$f'(x) = h'(g(x))g'(x).$$

How do you calculate $h'(g(x))$?

1. Write down the function $h(x)$

2. Differentiate $h(x)$ with respect to $x$ (don’t worry about $g(x)$).

3. Evaluate $h'(x)$ at $g(x)$. 
Example 3.1 Consider the function \( f(x) = (1 + \cos x)^{-4} \). We notice that the “inside” is \( g(x) = 1 + \cos x \), and the outside function is \( h(x) = x^{-4} \). Then

\[
h(g(x)) = (g(x))^{-4} = (1 + \cos x)^{-4} = f(x)
\]

so we may apply the chain rule. We first calculate \( g'(x) \):

\[
g'(x) = 0 + -\sin x = -\sin x.
\]

Now we calculate \( h'(g(x)) \) using the steps outlined above.

1. Write down \( h(x) \): \( h(x) = x^{-4} \).

2. Differentiate \( h(x) \) with respect to \( x \): \( h'(x) = -4x^{-5} \), using the power rule.

3. Evaluate \( h'(x) \) at \( g(x) \): \( h'(g(x)) = -4(g(x))^{-5} = -4(1 + \cos x)^{-5} \).

Lastly we use the chain rule:

\[
f'(x) = h'(g(x))g'(x) = -4(1 + \cos x)^{-5}(-\sin x) = +4(\sin x)(1 + \cos x)^{-5}
\]

Notice that the final answer is just using the power rule to \( (1 + \cos x)^{-4} \) times the “derivative of the inside”, which is \(-\sin x\).

Example 3.2 You Try It

1. Express \( f(x) = \sin(2x^2 - 1) \) as the composition of two functions. What is \( g(x) \) and what is \( h(x) \)? Check that \( h(g(x)) = f(x) \).

2. Find \( f'(x) \) using the chain rule.

3. Do Exercises 7, 9, 12, 13, 14, 17, 19, 23, 27, 30 on page 195 using the techniques outlined above. Check your answer in the back of the book.
4 Another look at the chain rule

You can also do the chain rule by applying the formula:

\[
\frac{df}{dx} = \frac{df}{du} \frac{du}{dx}.
\]

Why is this the same as above? The answer is very tricky!! Part of the trick is just notational. Let \( u = g(x) \). Then \( f(x) = h(g(x)) = h(u) \). The original chain rule says:

\[
\frac{df}{dx} = h'(g(x))g'(x).
\]

The term \( g'(x) \) is clearly the same as \( \frac{dg}{dx} = \frac{du}{dx} \) since \( u = g(x) \). Let’s take a closer look at \( h'(g(x)) \). Here’s the first trick: The functions \( h(x) \) and \( h(u) \) are really the same thing except there’s a variable change. Similarly, \( h'(x) = \frac{dh}{dx} \) is describing the same function as \( h'(u) = \frac{dh}{du} \), except the variable \( u \) is used instead of \( x \). Then \( h'(g(x)) \) (where the derivative is done with respect to \( x \), and then \( g(x) \) is stuck in for \( x \), is the same as \( \frac{dh}{du} \), take the derivative of \( h \) as a function of \( u \) and then evaluates at \( g(x) \). However, \( u = g(x) \) means that you could can just consider \( \frac{dh}{du} \) already as a function of \( u \) and you don’t need to stick in \( g(x) \). In practice, in order to evaluate \( \frac{dh}{du} \), we do actually plug in \( g(x) \) for \( u \), since we want it as a function of \( x \) in the end. This would lead us to the conclusion that

\[
\frac{df}{dx} = \frac{dh}{du} \frac{du}{dx}.
\]

So here’s the second trick: If we consider \( f \) as a function of \( u \), we mean that \( f = h(u) \)! This is a subtle notational issue. To illustrate this point, consider \( f(x) = ((x^2) + 1)^3 \). On the one hand, when we write \( f(x) \) to mean “stick \( x \) into the formula”. In this light, if we write \( f(u) \), we would mean \( f(u) = ((x^2) + 1)^3 \). But considering \( f \) as a function of \( u \) is different: let \( u = x^2 + 1 \). Then by saying “\( f \) as a function of \( u \),” we really mean the function \( u^3 \). This is exactly the “outside function”, or what we have been calling \( h(u) \)! Therefore, \( \frac{dh}{du} = \frac{df}{du} \), and sticking that into the equation above, we obtain

\[
\frac{df}{dx} = \frac{df}{du} \frac{du}{dx}.
\]
The main point in working problems using this, is that \( \frac{df}{du} \) means to think of \( f \) as a function of \( u \), where \( u \) is a convenient substitution that makes \( f \) look simpler. It is not actually \( f(u) \), but \( h(u) \), the outside function in our original discussion.

Example 4.1 Differentiate \( \cos(t^3 - 5t) \) using the second form of the chain rule.

Solution: We use the formula (with \( t \) replacing \( x \)):

\[
\frac{df}{dt} = \frac{df}{du} \frac{du}{dt}.
\]

Let \( u(t) = t^3 - 5t \). Then \( f = \cos u \) and \( \frac{df}{du} = -\sin u = -\sin(t^3 - 5t) \).

Also, \( \frac{du}{dt} = 3t^2 - 5 \). The final answer is then

\[
\frac{df}{dt} = \frac{df}{du} \frac{du}{dt} = -\sin(t^3 - 5t)(3t^2 - 5) = -(3t^2 - 5)\sin(t^3 - 5t).
\]

5 Chain Rule and Parameterized Curves

You may be wondering: what does the chain rule and differentiating functions have to do with parametric curves? The main answer is the following: if you are given a curve in parametric equations, how do you find the tangent line to a curve? In particular, how do you find its slope?

Remark 2 If you have a curve described by the graph of a function \( f(x) \), then the slope of the tangent line at a point \( P(x_0, y_0) \) is \( f'(x_0) \) (the derivative \( f'(x) \), evaluated at \( P \)).

Remark 3 If you have a curve described by parametric equations \( x = f(t) \) and \( y = g(t) \), then the slope of the tangent line at a point \( P(x_0, y_0) \) is given by \( \frac{dy}{dx} \), evaluated at \( P \).

So the real question boils down to: How do you calculate \( \frac{dy}{dx} \) for a parametric curve (a curve described by parametric equations)?

The answer is using the CHAIN RULE for parametric equations
Theorem 5.1 (The Chain Rule for Parametric Curves) Let \( x = f(t) \) and \( y = g(t) \) be parametric equations for a curve, where \( t \in D \), \( D \) a domain in \( \mathbb{R} \). Then

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dg/dt}{df/dt}.
\]

You may wonder: How is this the chain rule? Multiply both sides of the equation by \( dx/dt \) and get

\[
\frac{dy}{dx} \cdot dx/dt = \frac{dy}{dt}
\]

which is the chain rule for calculating \( \frac{dy}{dt} \). This seems weird: we have \( y \) as a function of \( t \) (mainly \( g(t) \)). So what does the left hand side of the equation mean? We are just applying a trick here. We’d LIKE to have \( \frac{dy}{dx} \) even though we have no explicit expression for \( y \) in terms of \( x \), as we do with Cartesian functions of the form \( y = f(x) \). So we PRETEND that \( y \) were a composition with \( x(t) \). If it were, i.e. \( y(t) = h(x(t)) \), then by the same reasoning above, we’d obtain that

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}
\]

where \( x \) is playing the role that \( u \) does in the previous section. Now it happens that we know \( \frac{dy}{dt} \) and \( \frac{dx}{dt} \), so we may solve for \( \frac{dy}{dx} \) and it doesn’t matter that we never actually wrote \( y \) down as a composition of functions!!

Let’s try to use this formula:

Example 5.1 Let \( x = \sin t \), \( y = \sqrt{5} \sin t \). Find the slope of the tangent curve at \( t = 2\pi/3 \). Then find the equation of the line through this point tangent to the curve.

Solution: We use the formula:

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.
\]

By direct calculation, \( \frac{dy}{dt} = \sqrt{5} \cos t \) and \( \frac{dx}{dt} = \cos t \). The ratio is then

\[
\frac{dy}{dx} = \sqrt{5}
\]
We evaluate at \( t = \frac{2\pi}{3} \), but there is no \( t \)-dependence, so the answer is \( \sqrt{5} \). To find the line through the curve, we use

\[
y = mx + b = \sqrt{5}x + b.
\]

Using the point \((\sin 2\frac{2\pi}{3}, \sqrt{5}\sin 2\frac{2\pi}{3}) = (\frac{\sqrt{3}}{2}, \frac{\sqrt{15}}{2})\) we solve for \( b \):

\[
\frac{\sqrt{15}}{2} = \sqrt{5}\frac{\sqrt{3}}{2} + b
\]

so \( b = 0 \) and the equation of the line is just \( y = \sqrt{5}x \).

**Example 5.2 You Try It:**

1. Let \( x = \cos^2 t \) and \( y = \sin t \), \( t \in [0, 2\pi) \). Find the slope of the tangent line to the curve at \( t = \frac{\pi}{4} \) by finding

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.
\]

and then evaluating at \( t = \frac{\pi}{4} \). Then find the equation of the line tangent to the curve at this point.

2. Do problems 33, 35, 37, 39 on p. 195.