Introduction to nonlinear FEM

Most physical problems are inherently nonlinear in nature. Such nonlinearities arise due to a non-linear strain-displacement (Geometric) relation or due to a non-linear stress-strain (Material) relationship. Geometric nonlinearities occur due to large displacements, large strains, large rotations and so on. Material nonlinearities occur when the stress-strain (Constitutive Law) or force-displacement relationship is not linear, or when material properties change with applied loads. For example, the Young's modulus parameter may vary with displacement i.e. \( E = E(u) \).
Let us consider the axial deformation of a bar with distributed load \( q(x) \) as before. This time let us assume a non-linear stress-strain relationship of the form

\[
\sigma(x) = F(u) E(x)
\]

\[
= F(u) \frac{du}{dx}
\]

Substituting this into the governing differential equation for the bar is

\[
A \frac{d\sigma}{dx} + q(x) = 0
\]

\[
\Rightarrow A \frac{d}{dx} \left[ F(u) \frac{du}{dx} \right] + q(x) = 0
\]

To obtain the weak-variational form, we multiply this equation by a test function \( w(x) \) over the element and integrate by parts. This yields
\[
\int_{0}^{L} A \frac{d}{dx} \left[ E(u) \frac{du}{dx} \right] v(x) \, dx = -\int_{0}^{L} q(x) v(x) \, dx
\]

\[
\Rightarrow A E(u) \frac{du}{dx} \left[ \int_{0}^{L} v(x) \, dx \right] - \int_{0}^{L} A E(u) \frac{du}{dx} \frac{dv}{dx} \, dx = -\int_{0}^{L} q(x) v(x) \, dx
\]

Applying the boundary conditions used as in the linear case, namely,

\[U(0) = 0\]

\[A E(u) \frac{du}{dx}(L) = R\]

where \( R \) was a concentrated load at \( x = L \),

we get,

\[
\int_{0}^{L} A E(u) \frac{du}{dx} \frac{dv}{dx} \, dx = \int_{0}^{L} q(x) v(x) \, dx + R \, v(L)
\]
As before we consider a finite dimensional approximation $u_h(x)$ of the continuous solution $u(x)$ such that, our problem becomes: Find $u_h(x) \in V_N$ such that

$$
\int_0^L A E(u_h) \frac{du_h}{dx} \frac{dv}{dx} \, dx = \int_0^L v(x) u(x) \, dx + R v(L)
$$

for all $v(x) \in V_N$ (= finite dimensional space).

Supposing that $V_N = \text{Span} \{ \phi_1(x), \phi_2(x), \ldots, \phi_N(x) \}$

where $\phi_i(x)$ is a basis function for $V_N$, then we can write, $u_h(x) = \sum_{i=1}^N c_i \phi_i(x)$ and $v(x) = \phi_j(x)$ for $j = 1 \ldots N$ using this our discrete variational form becomes:

$$
\int_0^L A E \left( \sum_{i=1}^N c_i \phi_i \right) \frac{d}{dx} \left( \sum_{i=1}^N c_i \phi_i \right) \frac{d}{dx} \phi_j \, dx = \int_0^L v(x) \phi_j(x) \, dx + R \phi_j(L)
$$

for $j = 1 \ldots N$. 

**Finite Element Discretization:**
This can be rewritten as:

\[
\sum_{i=1}^{N} c_i \int_0^L A E(u_i(x)) \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} = \int_0^L V(x) \phi_j(x) dx + R \phi_j(L) \quad j = 1, 2, \ldots, N
\]

This can be written in matrix form as

\[
K(\mathbf{\phi}) \mathbf{\phi} = F
\]

where

\[
K_{ij} = A \int_0^L F \left( \sum_{i=1}^{N} c_i \phi_i(x) \right) \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} dx
\]

\[
F_j = \int_0^L V(x) \phi_j(x) dx + R \phi_j(L)
\]

Notice that solving \((*)\) requires us to solve a nonlinear system of equations and therefore cannot be solved by direct methods.
Let us assume  

\[ F(u) = F_0 \left(1 - \frac{du}{dx}\right) \]

In terms of the coefficients we have,

\[ F(U_n) = F_0 \left(1 - \frac{d}{dx} u_h \right) \]

\[ = F_0 \left(1 - \sum_{k=1}^{L} c_k \frac{d\Phi_k}{dx} \right) \]

Then the entries of the stiffness matrix become,

\[ K_{ij} = AE_0 \int_{0}^{L} (1 - \sum_{k=1}^{L} c_k \frac{d\Phi_k}{dx}) \frac{d\Phi_i}{dx} \frac{d\Phi_j}{dx} \, dx \]

As in the linear case we will express the entries of the global stiffness matrix in terms of local stiffness matrices. For this we partition the domain \([0, L]\) into \(M\) subintervals and then we can express

\[ K_{ij} = \sum_{e=1}^{M} K_{ij}^{(e)} \]

where

\[ K_{ij}^{(e)} = AE_0 \int_{x_e}^{x_{e+1}} (1 - \sum_{k=1}^{L} c_k^{(e)} \frac{d\Phi_k^{(e)}}{dx}) \frac{d\Phi_i^{(e)}}{dx} \frac{d\Phi_j^{(e)}}{dx} \, dx \]
Each of these local elements \([x_e, x_{e+1}]\) are mapped using linear transformations to \([-1, 1]\) as in the linear case to perform the integral computation. The local basis functions over this reference element \([-1, 1]\) is given by

\[
N_1(\xi) = \frac{1-\xi}{2} \quad N_2(\xi) = \frac{1+\xi}{2}
\]

Let us now consider the simple case of the axial bar divided into two elements (i.e., with three nodes \(x_1, x_2, x_3\)). We can then write the element stiffness matrix over \([x_e, x_{e+1}]\) \(c=1,2\) as:

\[
K_{ij}^{(e)} = AE_0 \int_{x_e}^{x_{e+1}} \left(1 - C_1 \frac{d\phi_1^e}{dx} - C_2 \frac{d\phi_2^e}{dx}\right) \frac{d\phi_j^e}{dx} \frac{d\phi_i^e}{dx} dx
\]

which can be computed using the transformation between \([x_e, x_{e+1}]\) and \([-1, 1]\) as before.
The local stiffness matrix work out to

\[
K^{(e)} = \frac{AE_0}{h_e^2} \left( h + c_1^e - c_2^e \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

where \( c_1^e, c_2^e \) correspond to unknown coefficients over each element. For the simple example of two elements \( c_1 = c_1^1, c_2 = c_2^1, c_1 = c_2^2, c_2 = c_3 \).

Next step is to assemble the local stiffness matrix to yield the following global stiffness matrix (for uniform grid with \( h_e = h \)):

\[
K^{(G)} = \frac{AE_0}{h^2} \begin{bmatrix}
  h + c_1 - c_2 & -(h + c_1 - c_2) & 0 \\
  -(h + c_1 - c_2) & (2h + c_1 - c_3) & -(h + c_2 - c_3) \\
  0 & -(h + c_2 - c_3) & (h + c_2 - c_3)
\end{bmatrix}
\]

Note that after assembly one must reduce
(On condense) the global stiffness matrix to account
for the boundary condition at \( x = 0 \) which
yields the following reduced stiffness matrix:

\[
\tilde{K}(\vec{c}) = \frac{AE_0}{h^2} \begin{bmatrix}
2h - c_3 & -(h + c_2 - c_3) \\
-(h + c_2 - c_3) & h + c_2 - c_3
\end{bmatrix}
\]

It must be noted that one must perform
similar computations on the right hand side
as in the linear case. Suppose we consider
\( q(x) = 0 \) (for simplicity) then we obtain the
following reduced nonlinear system:

\[
\frac{AE_0}{h^2} \begin{bmatrix}
2h - c_3 & -(h + c_2 - c_3) \\
-(h + c_2 - c_3) & h + c_2 - c_3
\end{bmatrix} \begin{bmatrix}
c_2 \\
c_3
\end{bmatrix} = \begin{bmatrix}
0 \\
R
\end{bmatrix}
\]

on \( \tilde{K}(\vec{c}) \vec{c} = F \)
To solve this we consider the residual system

\[
\vec{R}(\vec{C}) = \hat{\vec{K}}(\vec{C}) \vec{C} - \vec{F} = 0
\]

and employ the Newton-Raphson method. Starting with an initial guess \( \vec{C}_0 = [c_2, c_3] \), the Newton-Raphson iterations yield:

\[
\vec{C}_{n+1} = \vec{C}_n - T^{-1}(\vec{C}_n) \vec{R}(\vec{C}_n)
\]

where \( T(\vec{C}_n) \) is the tangent stiffness matrix given by:

\[
T(\vec{C}_n) := \frac{\partial \vec{R}(\vec{C}_n)}{\partial \vec{C}_n}
\]

which can be computed as:

\[
T(\vec{C}_n) = \frac{\Lambda \epsilon_0}{h^2} \begin{bmatrix}
2(h - c_3) & -h - 2(c_2 - c_3) \\
-h - 2(c_2 - c_3) & h + 2(c_2 - c_3)
\end{bmatrix}
\]
The inverse of \( T(\mathbf{c}_n) \) can be computed to be:

\[
T^{-1}(\mathbf{c}_n) = \frac{h^2}{AE_0 (h - 2c_2)} \begin{bmatrix} 1 & 1 \\ 1 & \frac{2h - 2c_3}{h + 2c_2 - 2c_3} \end{bmatrix}
\]

To summarize one must solve the following iteratively:

\[
\begin{bmatrix} c_{z_n} \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} - T^{-1}(\mathbf{c}_n) \cdot \tilde{R}(\mathbf{c}_n)
\]

where \( T^{-1}(\mathbf{c}_n) \) is given above and

\[
\tilde{R}(\mathbf{c}_n) = \tilde{K}(\mathbf{c}_n) \mathbf{c}_n - \tilde{F}
\]

\[
= \frac{A E_0}{h^2} \begin{bmatrix} 2h - c_3 & -(h + c_2 - c_3) \\ -(h + c_2 - c_3) & h + c_2 - c_3 \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} - \begin{bmatrix} 0 \\ R \end{bmatrix}
\]
Let \(A = 0.01 \text{ m}^2\), \(L = 2 \text{ m}\), \(E_0 = 100 \text{ MPa} = 100 \times 10^6 \text{ Pa}\)
\(R = 100 \text{ kN} = 100 \times 10^3 \text{ N}\). Since there are only two elements \(h = 1 \text{ m}\). Starting with a zero guess
\[
\begin{bmatrix}
\mathbf{C}_0 \\
\mathbf{C}_3
\end{bmatrix}_0 = \begin{bmatrix}
0 \\
0
\end{bmatrix} \Rightarrow \hat{\mathbf{K}}(\mathbf{C}_0) = 10^6 \begin{bmatrix}
2 & -1 \\
-1 & 1
\end{bmatrix} = T(\mathbf{C}_0)
\]

Then the Newton iterations yield:
\[
\begin{bmatrix}
\mathbf{C}_1 \\
\mathbf{C}_2
\end{bmatrix}_1 = \begin{bmatrix}
0.1 \\
0.2
\end{bmatrix} \hat{\mathbf{K}}(\mathbf{C}_1) = 10^6 \begin{bmatrix}
1.8 & -0.9 \\
-0.9 & 0.9
\end{bmatrix} \Rightarrow T(\mathbf{C}_1) = 10^6 \begin{bmatrix}
1.6 & -0.8 \\
-0.8 & 0.8
\end{bmatrix}
\]
\[
\begin{bmatrix}
\mathbf{C}_2 \\
\mathbf{C}_3
\end{bmatrix}_2 = \begin{bmatrix}
0.1125 \\
0.2250
\end{bmatrix} \hat{\mathbf{K}}(\mathbf{C}_2) = 10^6 \begin{bmatrix}
1.775 & -0.8875 \\
-0.8875 & 0.8875
\end{bmatrix} \Rightarrow T(\mathbf{C}_2) = 10^6 \begin{bmatrix}
1.55 & -0.775 \\
-0.775 & 0.775
\end{bmatrix}
\]
\[
\begin{bmatrix}
\mathbf{C}_3 \\
\mathbf{C}_4
\end{bmatrix}_3 = \begin{bmatrix}
0.1127 \\
0.2254
\end{bmatrix} \hat{\mathbf{K}}(\mathbf{C}_3) = 10^6 \begin{bmatrix}
1.7746 & -0.8873 \\
-0.8873 & 0.8873
\end{bmatrix} \Rightarrow T(\mathbf{C}_3) = 10^6 \begin{bmatrix}
1.5492 & -0.7746 \\
-0.7746 & 0.7746
\end{bmatrix}
\]
\[
\begin{bmatrix}
\mathbf{C}_4
\end{bmatrix}_4 = \begin{bmatrix}
0.1127 \\
0.2254
\end{bmatrix}
\]
We stop here as the solution seems to have converged. We use this solution for post-processing.