Consider the Boundary Value Problem:

\[-U''(x) + U(x) = f(x)\]

\[U(0) = 0\]

\[U(1) = 0\]

Let us define the space \(V\) as follows:

\[V = \left\{ u \in \mathbb{E} : \int_{0}^{1} (u'(x))^2 + (u(x))^2 < \alpha \text{ and } u(0) = u(1) = 0 \right\}\]

Let us also define the norm on the space to be

\[\|u\|_V = \left( \int_{0}^{1} (u'(x))^2 + (u(x))^2 \right)^{\frac{1}{2}}\]

Note that the weak variational problem for the BVP becomes: Find \(u(x) \in V\) so:

\[\int_{0}^{1} u'(x) v'(x) dx + \int_{0}^{1} u(x) v(x) dx = \int_{0}^{1} f(x) v(x) dx + \int_{0}^{1} 0 v(x) dx\]
Let us define a bilinear functional

\[ a(u, v) := \int_0^1 u'(x) v'(x) \, dx + \int_0^1 u(x) v(x) \, dx \]

and a linear functional:

\[ F(v) := \int_0^1 f(x) v(x) \, dx \]

Our problem then becomes: Find \( u(x) \in V \) that satisfies \( a(u, v) = F(v) \) for \( v(x) \in V \).

For this problem to have an unique solution we have the following theorem:

**Lax–Milgram Theorem**

(A) Let \( V \) be a Hilbert space with scalar product \( (\cdot, \cdot)_V \) and norm \( \| \cdot \|_V = (\cdot, \cdot)_V^{\frac{1}{2}} \).

(B) Let \( a(\cdot, \cdot) \) be a symmetric bilinear form on \( V \times V \) such that:
(i) \( a(\cdot, \cdot) \) is continuous (bounded) that is for a constant \( \gamma > 0 \) there exists:
\[
|a(v, w)| \leq \gamma \|v\| \|w\| + \gamma, v, w \in V
\]

(ii) \( a(\cdot, \cdot) \) is \( V \)-elliptic (coercive) that is for \( \alpha > 0 \) there exists:
\[
a(v, v) \geq \alpha \|v\|^2 \quad \forall v \in V
\]

(c) Finally, suppose that there is a continuous (bounded) linear form \( F \) on \( V \) i.e.
for a constant \( \lambda > 0 \) there exists:
\[
|F(v)| \leq \lambda \|v\|, \quad \forall v \in V
\]
Then there exists a unique solution \( u(x) \in V \) that satisfies:
\[
a(u, v) = F(v) + \forall v \in V
\]
For the BVP considered note that,

\[ |a(v, w)| = \left| \int_0^1 v'(x) w'(x) \, dx + \int_0^1 v(x) w(x) \, dx \right| \]

**Triangle Inequality**

\[ \leq \left| \int_0^1 v'(x) w'(x) \, dx \right| + \left| \int_0^1 v(x) w(x) \, dx \right| \]

**Cauchy-Schwarz Inequality**

\[ \leq \left( \int_0^1 (v'(x))^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^1 (w'(x))^2 \, dx \right)^{\frac{1}{2}} + \left( \int_0^1 (v(x))^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^1 (w(x))^2 \, dx \right)^{\frac{1}{2}} \]

\[ \leq 2 \left( \int_0^1 (v'(x))^2 + (v(x))^2 \right)^{\frac{1}{2}} \left( \int_0^1 (w'(x))^2 + (w(x))^2 \right)^{\frac{1}{2}} \]

\[ = 2 \| v \|_1 \| w \|_1 \]

Hence we have shown that \( a(v, w) \) is bounded.

Also \( a(v, v) = \int_0^1 \left( (v'(x))^2 + (v(x))^2 \right) \, dx \)

\[ \geq C \int_0^1 \left( (v'(x))^2 + (v(x))^2 \right) \, dx \quad \text{with} \quad C = 1 \]

\[ = C \| v \|_1^2 \quad \text{(Coercivity)} \]
Finally we also note that

$$ \left| \int_0^1 f(x) \psi(x) \, dx \right| \leq (\int_0^1 (f(x))^2 \, dx)^{1/2} \left( \int_0^1 (\psi(x))^2 \, dx \right)^{1/2} $$

$$ \leq \| f \|_2 \| \psi \|_2 $$

Hence choosing $\lambda = \| f \|_2$ we also note that $F(\psi)$ is bounded. Therefore we can now employ the Lax-Milgram theorem which yields the result that the solution to the BVP given is unique.
To determine an approximate solution, we consider a finite dimensional space \( V_N \subset V \) and we look for \( u_h(x) \in V_N \), which also satisfies:

\[
\alpha(u_h, v) = F(v) + v \in V_h
\]

Since the exact solution satisfies \((u(x) \in V)\)

\[
\alpha(u, v) = F(v) + v \in V_h
\]

Subtracting these equations, we get,

\[
\alpha(u - u_h, v) = 0 + v \in V_h
\]

which is an important orthogonality relationship satisfied by the approximate solution \( u_h(x) \). Let us now try to use this orthogonality relation to prove the following basic error estimate.
**Theorem:** \( \| u - u_h \| \leq \frac{\alpha}{\gamma} \| u - w_h \| + w_h \in V_h \)

Let \( w_h \in V_h \), \( V_h \subset V \) such that \( V_h = u_h - w_h \).

From coercivity we have,

\[
\alpha \| u - u_h \|^2 \leq \alpha (u - u_h, u - u_h)
\]

\[
= \alpha (u - u_h, u - u_h) + \alpha(u - u_h, u - w_h)
\]

**Using Orthogonality**

\[
= \alpha (u - u_h, u - w_h)
\]

**Boundedness**

\[
\leq \gamma \| u - u_h \| \| u - w_h \|
\]

\[
\Rightarrow \| u - u_h \| \leq \frac{\alpha}{\gamma} \| u - w_h \| + w_h \in V_h
\]