Analysis of the Finite Element Method
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Padmanabhan Seshaiyer
Mathematical Sciences
George Mason University
Email: pseshaiy@gmu.edu
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\[ A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \]
Strong $\iff$ Weak $\iff$ Minimization

- **Strong Form**: Let $f \in C^0([0,1])$. Find $u \in C^2([0,1])$
  
  \[
  \begin{align*}
  -u''(x) &= f(x) \quad \text{in } (0,1) \\
  u(0) &= 0 \quad u'(1) = 0
  \end{align*}
  \]

- **Weak Form**: Find $u \in V$ such that: $a(u, v) = (f, v)$ for all $v \in V$ where $V = \left\{ v \in L^2(0,1) : a(v, v) < \infty \text{ and } v(0) = 0 \right\}$ we have,
  \[
  a(u, v) = \int_0^1 u'(x) v'(x) \, dx \quad \text{and} \quad (f, v) = \int_0^1 f(x) v(x) \, dx
  \]

- **Minimization Problem**: Find $u \in V$ such that:
  \[
  M(u) \leq M(v) \quad \text{for all } v \in V
  \]
Ritz-Galerkin Approximation

• Let $S \subset V$ be any finite dimensional subspace.

$$S = \text{span}\{\phi_1(x), \phi_2(x), \ldots, \phi_N(x)\}$$

• Weak form on finite dimensional space:
  - Find $u_S \in S$ such that: $a(u_S, v) = \langle f, v \rangle$ for all $v \in V$

• Choosing $v = \phi_i(x)$ for any $i=1..n$ and $u_S(x) = \sum_{j=1}^{N} c_j \phi_j(x)$ we have

- Find $\tilde{c} = \{c_j\}$ that satisfies the matrix system: $K \tilde{c} = \tilde{f}$

where,

$$K_{ij} = a(\phi_j, \phi_i) \quad i, j = 1..n$$

$$f_i = \langle f, \phi_i \rangle \quad i = 1..n$$
Linear Basis Functions

• Partition [0, 1]

\[ 0 = x_0 < x_1 < ... < x_{j-1} < x_j < x_{j+1} < ... < x_M = 1 \]

\[ h_j = x_j - x_{j-1} \] (Non-uniform grid)

\[ \Delta x = h_j \] (Uniform grid)

• For \( j = 1, \ldots, n \), let

\[ \phi_j(x) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \]

\[ u_S(x_i) = \sum_{j=1}^{N} c_j \phi_j(x_i) = c_i \]
Estimates for $\| u - u_s \|_V$

- **Theorem:** If $u''$ is continuous then,

\[ \| u - u_s \|_V \leq c h \max |u''(x)| \]

- **Key steps in proof**
  - Consider the piecewise linear interpolant $u_I$ and prove
    \[ \max |u(x) - u_I(x)| \leq \frac{h^2}{8} \max |u''(x)| \]
    \[ \max |u'(x) - u'_I(x)| \leq h \max |u''(x)| \]
  - Estimate $\| u - u_s \|_V$ using the above results to prove theorem.
Need a better result

• The factor “h” in the estimate reflects the observed rate of convergence as mesh is refined.

• The imperfection is $\max |u''(x)|$. It is not satisfactory to have to assume that $u''$ is continuous or even that it is bounded when we require a result in the energy norm.

• So we look for an estimate that only assumes that $u''$ has finite energy in the $L^2$ norm.
Estimates for $\| u - u_s \|_{H^1(0,1)}$

- **Theorem:** If $u'' \in L^2$ then,

  $$\| u - u_s \|_{H^1(0,1)} \leq \frac{h}{\pi} \left( 1 + \frac{h^2}{\pi^2} \right)^{\frac{1}{2}} \| u''(x) \|_{L^2(0,1)}$$

- **Key steps in proof**
  - Consider the piecewise linear interpolant $u_I$ and prove

    $$\| u - u_I \|_{L^2(0,1)} \leq \frac{h^2}{\pi^2} \| u'' \|_{L^2(0,1)}$$

    $$\| u' - u'_I \|_{L^2(0,1)} \leq \frac{h}{\pi} \| u'' \|_{L^2(0,1)}$$

  - Estimate $\| u - u_s \|_{H^1(0,1)}$ using the above results to prove theorem.