\[ u_t + uu_x = u_{xx} \quad 0 < x < 1, \quad t > 0 \]
\[ u(0, t) = u(1, t) = 0 \quad t > 0 \]
\[ u(x, 0) = e(x) \quad 0 < x < 1 \]

Let \( v : [0, 1] \rightarrow \mathbb{R} \)
\[ x \rightarrow v(x) \]

\[ u_t v + \frac{1}{2} (u^2)_x v = u_{xx} v \]
\[ u(0, t) = u(1, t) = 0 \]
\[ u(x, 0) = e(x) \]

\[ \int_0^1 u_t v dx + \frac{1}{2} \int_0^1 (u^2)_x v dx = \int_0^1 u_{xx} v dx \]
\[ u(0, t) = u(1, t) = 0 \]
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Integration by parts 8
\[ \int_0^1 u_t v dx + \frac{1}{2} \int_0^1 (u^2)_x v dx = - \int_0^1 u_x v_x dx + u_x v \bigg|_0^1 \]

Let's assume that
\[ v = 0 \] at \( x = 0 \) and \( x = 1 \) then
\[
\int_0^1 u_t v \, dx + \frac{1}{2} \int_0^1 (u^2)_x v \, dx = -\int_0^1 u_x v_x \, dx \tag{2}
\]

So, if \( u(x, t) \) satisfies (1) then \( u(x, t) \) satisfies (2) for every \( V(x) \) with \( V(0) = V(1) = 0 \).

Conversely, if \( u(x, t) \) with \( u(0, t) = u(1, t) = 0 \) satisfies (2) for every \( V(x) \) with \( V(0) = V(1) = 0 \) then
\[
u_t + uu_x = u_{xx}, \quad 0 < x < 1, \quad t > 0
\]
\[
u(0, t) = u(1, t) = 0, \quad t > 0
\]

for each fixed \( t \).

Let \( H := \{ V : [0, 1] \to \mathbb{R} \} \)

Continuous and “a little more,” and \( v(0) = v(1) = 0 \) (fix \( t \)) we are searching for \( u(x, t) \in H \) such that (2) is satisfied for every \( V \in H \).

But \( H \) is infinite dimensional, \( \text{span} \{ \sin(n \pi x) \} \) for \( n = 1, 2, 3, \ldots \) linearly independent. (Not good for numerical purposes.) so we wish to restrict our search space to a finite dimensional one \( V \in H \).

For solution

\[
x_1, x_2, x_3, \ldots, x_n, \quad 0 = x_0 < x_1 < \ldots < x_n = 1 = x_{n+1}
\]
$V := \{ v : [a, b] \rightarrow \mathbb{R} : v \text{ is continuous and }$

\begin{align*}
V|_{[x_k, x_{k+1}]} & \text{ is linear for } k = 0, \ldots, n \text{ and } V(0) = V(1) = 0 \}\}
\end{align*}$

$V$ is finite dimensional since $B$

form a basis for $V$.

$$V_{x_k}(x) = \begin{cases} 
\frac{x-x_k}{x_k-x_{k-1}} & x \in [x_{k-1}, x_k) \\
\frac{x_{k+1}-x}{x_{k+1}-x_k} & x \in (x_k, x_{k+1}] \\
0 & \text{otherwise} 
\end{cases}$$

$k = 1, \ldots, n$
Find \( u(x,t) \) such that for each fixed \( t \), \( u(\mathbf{x}, t) \in V \) and \( \mathbf{u} \) such that for each \( \mathbf{v} \in V \), \( u(\mathbf{x}, t) \) satisfies
\[
\int_0^1 u_t \mathbf{v} \, dx + \frac{1}{2} \int_0^1 (u^2)_x \mathbf{v} \, dx = -\int_0^1 u_x \mathbf{v}_x \, dx \quad \forall t > 0
\]
\[
\int_0^1 u(x,0) \mathbf{v}(\mathbf{x}) \, dx = \int_0^1 \mathbf{c}(\mathbf{x}) \mathbf{v}(\mathbf{x}) \, dx
\]

\( u(\mathbf{x}, t) \in V \Rightarrow \)
\[
u(\mathbf{x}, t) = \sum_{i=1}^{N} c_i(t) \mathbf{v}_i(\mathbf{x})
\]
\[
\Rightarrow u_t = \sum_{i=1}^{N} \dot{c}_i(t) \mathbf{v}_i(\mathbf{x})
\]

\[
[u(x_j, t)]^2 = \left[ \sum_{i=1}^{N} c_i(t) \mathbf{v}_i(x_j) \right]^2
\]
\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} c_i(t) c_j(t) \mathbf{v}_i(x_j) \mathbf{v}_j(x_j)
\]

Let's assume that
\[
[u(x_j, t)]^2 = \sum_{i=1}^{N} c_i(t) \mathbf{v}_i(\mathbf{x})
\]

\[
\sum_{i=1}^{N} \dot{c}_i(t) \int_0^1 \mathbf{v}_i \mathbf{v}_j \, dx + \frac{1}{2} \sum_{i=1}^{N} c_i(t) \int_0^1 \mathbf{v}_i \mathbf{v}_j \, dx = -\sum_{i=1}^{N} c_i(t) \int_0^1 \mathbf{v}_i \mathbf{v}_j \, dx
\]
\[
\sum_{i=1}^{N} c_i(0) \int_0^1 \mathbf{v}_i \mathbf{v}_j \, dx = \int_0^1 \mathbf{c}(\mathbf{x}) \mathbf{v}_j(\mathbf{x}) \, dx \quad j = 1, \ldots, N
\]
\[ w := \left[ \int_0^1 v_i v_j \right]_{i,j=1}^N \quad \text{and} \quad k := \left[ \int_0^1 v_i^* v_j \right]_{i,j=1}^N \]

\[ A := \left[ \int_0^1 v_i v_j^* \right]_{i,j=1}^N \]

\[ w = \frac{k}{\delta} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

\[ h = x_i - x_{i-1} \]

\[ A \in \mathbb{R}^{N \times N} \]

\[ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \]

\[ K = \frac{1}{2} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \]

\[ c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} \]
\[ M \dot{c} = -kc - \left( \frac{1}{2} \right) A (c^2) \]
\[ MC(0) = U_0 = \left[ \sum_{j=1}^{N} e_j(x) v_j(w) \right] \]

\( M \) is invertible (believe)

\[ \dot{c} = M^{-1} (-kc - \frac{1}{2} A (c^2)) \]
\[ c(0) = M^{-1} U_0 \]

\[ \Rightarrow C(t) = \left[ \begin{array}{c} C_1(t) \\ \vdots \\ C_N(t) \end{array} \right] \]

\[ \Rightarrow w(x,t) = \sum_{i=1}^{N} C_i(t) v_i(x) \]

→ a system of ordinary differential equations

\[ \left\{ \begin{array}{l}
M \dot{c} = -kc - \left( \frac{1}{2} \right) A (c^2) \\
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