1. (13.23) Let $A$ and $B$ be non-empty subsets of $\mathbb{R}$. Let $C = \{x + y : x \in A, y \in B\}$. Prove that if $A$ and $B$ have upper bounds, then $C$ has a least upper bound and $\sup C = \sup A + \sup B$.

**Proof:** Since $A$ and $B$ have upper bounds, they have least upper bounds or sups. Let $\alpha = \sup A$ and $\beta = \sup B$ and let $\gamma = \alpha + \beta$. Let $d \in C$, then $d = x + y$, for some $x \in A$ and $y \in B$. Thus $d = x + y \leq \alpha + \beta = \gamma$. Thus $\gamma$ is an upper bound of $C$. To show that it is the least upper bound, by Proposition 13.15 of the book, it will suffice to show that there is a sequence of elements in $C$ which converges to $\gamma$ (since $\gamma$ is already an upper bound for $C$). By the same proposition there is a sequence $\{a_n\}$ of elements of $A$ that converge to $\alpha$ and a sequence $\{b_n\}$ of elements of $B$ that converge to $\beta$. Then the sequence $\{a_n + b_n\}$ clearly converges to $\gamma$ (by a Theorem) and the elements $a_n + b_n$ are in $C$, so we are done.

2. (4.27) Determine if the following are true or false, given that $\{a_n\}$ is a convergent sequence.

(a) For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \epsilon$.

TRUE. Since $\{a_n\}$ must be a Cauchy sequence and the above is just the statement of the being a Cauchy sequence.

(b) There exists $n \in \mathbb{N}$ such that for all $\epsilon > 0$, $|x_{n+1} - x_n| < \epsilon$.

FALSE. There cannot be one $n$ that works for all epsilon. Let $a_n = 1/n$ which converges to 0. No matter how large we make $n$ we can pick $\epsilon = 1/2|a_{n+1} - a_n|$, so that $|x_{n+1} - x_n| < \epsilon$ is clearly false.

(c) There exist $\epsilon > 0$ such that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| < \epsilon$.

TRUE (this one caused the most problems).

Since the series converges, the set $\{a_n\}$ has both an upper and lower bound, which we denote by $M$ and $m$ respectively. Let $\epsilon = M - m + 1$, then clearly $|a_{n+1} - a_n| \leq M - m < \epsilon$ for all $n$.

(d) For all $n \in \mathbb{N}$, there exists $\epsilon > 0$ such that $|x_{n+1} - x_n| < \epsilon$.

TRUE. Here $\epsilon$ is dependent on your choice of $n$, so let $\epsilon = |x_{n+1} - x_n| + 1.$