1. Prove that $<$ is a transitive relation on $\mathbb{Q}$.

2. Prove that no set contains two of its lower bounds.

3. Let $S$ be a nonempty set of rational numbers. Prove that $x$ is a greatest lower bound of $S$ if and only if the following two properties hold:
   
   (a) $x$ is a lower bound of $S$.
   
   (b) For each positive real number $\varepsilon$, there exists a rational number $s \in S$ such that $x \leq s \leq x + \varepsilon$.

4. Give an example to show that two disjoint sets of rationals can have the same greatest lower bound.

5. Let $.a_1a_2a_3\ldots$ be an infinite decimal, and let

   $$\alpha = \{ q \in \mathbb{Q} : \text{there exists an } n \in \mathbb{N} \text{ such that } q < .a_1a_2\ldots a_n \}$$

   Prove that $\alpha$ is a real number.

   Let $b_1$ be the largest digit (i.e., a 0, 1, 2...9) such that $[b_1/10] \in \alpha$. Let $b_2$ be the largest digit such that $[(10b_1 + b_2)/10^2] \in \alpha$. For each $n \in \mathbb{N}$, define inductively $b_n$ to be the largest digit such that

   $$[(10^{n-1}b_1 + 10^{n-2}b_2 + \ldots + b_n)/10^n] \in \alpha$$

   Prove that $b_n = a_n$ for each $n \in \mathbb{N}$.