1.1 SYSTEMS OF LINEAR EQUATIONS

A linear equation:

\[ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \]

Examples:

\[ 4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3 \]

Not linear:

\[ 4x_1 - 5x_2 = x_1 x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6 \]

A system of linear equations (or a linear system):

A collection of one or more linear equations involving the same set of variables, say, \( x_1, \ldots, x_n \).

A solution of the system:

A list \((s_1, s_2, \ldots, s_n)\) of numbers that makes each equation in the system a true statement when the values \(s_1, \ldots, s_n\) are substituted for \(x_1, \ldots, x_n\), respectively.
EXAMPLE Two equations in two variables:

\[ x_1 - 2x_2 = -1 \]
\[ -x_1 + 3x_2 = 3 \]

A solution is a pair \((x_1, x_2)\) that lies on both lines.

Two other possibilities:

\[ x_1 - 2x_2 = -1 \]
\[ -x_1 + 2x_2 = 3 \]

\[ x_1 - 2x_2 = -1 \]
\[ -x_1 + 2x_2 = 1 \]

Basic Fact: A system of linear equations has either

(i) exactly one solution; or consistent
(ii) infinitely many solutions; or consistent
(iii) no solution. inconsistent
EXAMPLE Three equations in three variables. Each equation determines a plane in space.

i) The planes intersect in one point:

ii) The planes intersect in a line:

iii) There is no point common to all three planes:
The **solution set**: 

The set of all possible solutions of the system.

**Equivalent systems:**

Two linear systems with the same solution set.

**STRATEGY FOR SOLVING A SYSTEM:**

*Replace one system with an equivalent system that is easier to solve.*

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EXAMPLE

\[ x_1 - 2x_2 = -1 \]
\[ -x_1 + 3x_2 = 3 \]
\[ \rightarrow \]
\[ x_1 - 2x_2 = -1 \]
\[ x_2 = 2 \]

\[ \rightarrow \]
\[ x_1 = 3 \]
\[ x_2 = 2 \]
```
EXAMPLE

\[ x_1 - 2x_2 = -1 \]
\[ -x_1 + 3x_2 = 3 \]

\[ x_1 - 2x_2 = -1 \]
\[ x_2 = 2 \]

\[ x_1 = 3 \]
\[ x_2 = 2 \]
Matrix Notation

Coefficient matrix:

\[
\begin{bmatrix}
1 & -2 \\
-1 & 3
\end{bmatrix}
\]

Augmented matrix:

\[
\begin{bmatrix}
1 & -2 & -3 \\
-1 & 3 & 3
\end{bmatrix}
\]

Elementary row operations:

1. (Replacement) Add to one row a multiple of another row.
2. (Interchange) Interchange two rows
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Fact about Row Equivalence: If the augmented matrices of two linear systems are now equivalent, then the two systems have the same solution set.
EXAMPLE 1

\[
\begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    2x_2 - 8x_3 &= 8 \\
    -4x_1 + 5x_2 + 9x_3 &= -9 \\
\end{align*}
\]

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\begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    2x_2 - 8x_3 &= 8 \\
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\begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    x_2 - 4x_3 &= 4 \\
    -3x_2 + 13x_3 &= -9 \\
\end{align*}
\]

\[
\begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    x_2 - 4x_3 &= 4 \\
    x_3 &= 3 \\
\end{align*}
\]

\[
\begin{align*}
    x_1 - 2x_2 &= -3 \\
    x_2 &= 16 \\
    x_3 &= 3 \\
\end{align*}
\]

\[
\begin{align*}
    x_1 &= 29 \\
    x_2 &= 16 \\
    x_3 &= 3 \\
\end{align*}
\]

Solution is \((29, 16, 3)\)
Check:
Is \((29, 16, 3)\) a solution of the original system?

\[
x_1 - 2x_2 + x_3 = 0 \\
2x_2 - 8x_3 = 8 \\
-4x_1 + 5x_2 + 9x_3 = -9
\]

Substitute and compute:

\[
(29) - 2(16) + (3) = 29 - 32 + 3 = 0 \\
2(16) - 8(3) = 32 - 24 = 8 \\
-4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9
\]
TWO FUNDAMENTAL QUESTIONS

(1) Is the system consistent; that is, does a solution exist?
(2) If a solution exists, is it the only one; that is, is the solution unique?

EXAMPLE 2 Is this system consistent?

\[
\begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    2x_2 - 8x_3 &= 8 \\
    -4x_1 + 5x_2 + 9x_3 &= -9
\end{align*}
\]

Solution In Example 1 we row reduced this system to the “triangular” form:

\[
\begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    x_2 - 4x_3 &= 4 \\
    x_3 &= 3
\end{align*}
\]

Now we can see that a solution exists and it is unique. (Why?)
EXAMPLE 3’ Is this system consistent?

\[
\begin{align*}
3x_2 - 6x_3 &= 8 \\
x_1 - 2x_2 + 3x_3 &= -1 \\
5x_1 - 7x_2 + 9x_3 &= 0
\end{align*}
\]

\[
\begin{bmatrix}
0 & 3 & -6 & 8 \\
1 & -2 & 3 & -1 \\
5 & -7 & 9 & 0
\end{bmatrix}
\]

Solution Row operations on the augmented matrix:

\[
\begin{bmatrix}
1 & -2 & 3 & -1 \\
0 & 3 & -6 & 8 \\
5 & -7 & 9 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & -2 & 3 & -1 \\
0 & 3 & -6 & 8 \\
0 & 0 & 0 & -3
\end{bmatrix}
\]

To interpret this “triangular form” go back to equation notation:

\[
\begin{align*}
x_1 - 2x_2 + 3x_3 &= -1 \\
3x_2 - 6x_3 &= 8 \\
0 &= -3 \quad \leftarrow \text{Never true!}
\end{align*}
\]
EXAMPLE For what values of $h$ will the following system be consistent?

\[
\begin{align*}
3x_1 - 9x_2 &= 4 \\
-2x_1 + 6x_2 &= h
\end{align*}
\]

Solution Reduce the system to triangular form.

Add $2/3$ times row 1 to row 2:

\[
\begin{align*}
3x_1 - 9x_2 &= 4 \\
0x_1 + 0x_2 &= h + 8/3
\end{align*}
\]

only true if $h + 8/3 = 0$

The system is consistent precisely when $h = -8/3$.  
