June 2, 2015
Examine algorithms that use numerical approximation to solve problems
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- Convergence
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- Convergence
- Complexity
Class Overview

Examine algorithms that use numerical approximation to solve problems

- Convergence
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- Error
Class Overview

Examine algorithms that use numerical approximation to solve problems

- Convergence
- Complexity
- Error
- etc...
Class Overview

Textbook

NUMERICAL ANALYSIS
SECOND EDITION
TIMOTHY SAUER
Analysis done in MATLAB
CHAPTER 1: Solving Equations
Section 1.1

Assume a function $f(x)$
Section 1.1

Assume a function \( f(x) \)

- *Definition*: A root of \( f(x) \) is a solution such that \( f(r) = 0 \)
Assume a function $f(x)$

Definition: A root of $f(x)$ is a solution such that $f(r) = 0$

Goal: find root, $r$,
Assume a function $f(x)$

- **Definition**: A root of $f(x)$ is a solution such that $f(r) = 0$
- Goal: find root, $r$
  - Accurately
Section 1.1

Assume a function \( f(x) \)

- **Definition**: A root of \( f(x) \) is a solution such that \( f(r) = 0 \)
- **Goal**: find root, \( r \),
  - Accurately
  - As fast as possible (fewest operations)
Find root
Find root

Example: \( f(x) = 5x - 8 = 0 \rightarrow x = \frac{8}{5} \)
Section 1.1

Find root

- Example: \( f(x) = 5x - 8 = 0 \rightarrow x = \frac{8}{5} \)
- Example:
  \[ f(x) = x^2 + 2x - 3 = (x + 3)(x - 1) = 0 \rightarrow x = -3, 1 \]
Find root

- Example: \( f(x) = 5x - 8 = 0 \rightarrow x = \frac{8}{5} \)
- Example:
  \[
  f(x) = x^2 + 2x - 3 = (x + 3)(x - 1) = 0 \rightarrow x = -3, 1
  \]
- Example: \( f(x) = x^3 + x - 1 = 0 \rightarrow x = \text{???} \)
Section 1.1

\[ f(x) = x^3 + x - 1 \]

What we know:
\[ f(0) = -1 \]
\[ f(1) = 1 \]
Section 1.1

\[ f(x) = x^3 + x - 1 \]

What we know:
\[ f(0) = -1, \quad f(1) = 1 \]
Section 1.1

\[ f(x) = x^3 + x - 1 \]

What we know: \( f(0) = -1, f(1) = 1 \)
Assume a function \( f(x) \) defined on interval \([a, b]\).
Section 1.1

Assume a function $f(x)$ defined on interval $[a, b]$

*Theorem* - If $f(x)$ is continuous between $a$ and $b$ then $f$ has all values between $f(a)$ and $f(b)$ (Intermediate Value Theorem, or IVT)
Section 1.1

Assume a function \( f(x) \) defined on interval \([a, b]\)

**Theorem**- If \( f(x) \) is continuous between \( a \) and \( b \) then \( f \) has all values between \( f(a) \) and \( f(b) \) (Intermediate Value Theorem, or IVT)

**Corollary**- If \( f(x) \) is continuous and \( f(a)f(b) < 0 \), \( f \) has a root between \( a \) and \( b \)
Section 1.1

Let’s use IVT to our advantage
Section 1.1

Let’s use IVT to our advantage

**Bisection Method**

1. Select an interval \([a_i, b_i]\) such that \(f(a_i) f(b_i) < 0\).
2. Compute \(c_i = \frac{a_i + b_i}{2}\), i.e. the midpoint of the interval.
3. Compute \(f(c_i)\).
4. If \(f(a_i) f(c_i) < 0\), then \(b_{i+1} = c_i\) and our new interval is \([a_i, b_{i+1}]\).
5. If \(f(b_i) f(c_i) < 0\), then \(a_{i+1} = c_i\) and our new interval is \([a_{i+1}, b_i]\).
6. Repeat 2-5 until convergence.

Final interval \([a, b]\) contains a root; approximate root is \(\hat{r} = \frac{a + b}{2}\).
Let’s use IVT to our advantage

**Bisection Method**

1. Select an interval \([a_i, b_i]\) such that \(f(a_i)f(b_i) < 0\)
Let’s use IVT to our advantage

**Bisection Method**

1. Select an interval \([a_i, b_i]\) such that \(f(a_i)f(b_i) < 0\)
2. Compute \(c_i = (a_i + b_i)/2\), i.e. the midpoint of the interval
Section 1.1

Let’s use IVT to our advantage

**Bisection Method**

1. Select an interval \([a_i, b_i]\) such that \(f(a_i)f(b_i) < 0\)
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**Bisection Method**

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**Bisection Method**

1. Select an interval $[a_i, b_i]$ such that $f(a_i)f(b_i) < 0$
2. Compute $c_i = (a_i + b_i)/2$, i.e. the midpoint of the interval
3. Compute $f(c_i)$
4. If $f(a_i)f(c_i) < 0$, then $b_{i+1} = c_i$ and our new interval is $[a_i, b_{i+1}]$
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**Bisection Method**

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Final interval \([a, b]\) contains a root; approximate root is \(\hat{r} = (a + b)/2\).
Section 1.1

\[ f(x) = x^3 + x - 1 \]
Section 1.1

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Section 1.1

\[ f(x) = x^3 + x - 1 \]

For this function, Bisection Method converges to \( \hat{x} \approx 0.6821 \)
Section 1.1

How accurate is Bisection Method?

Assuming a starting interval $[a, b]$, after $n$ bisection steps the interval $[a_n, b_n]$ will have length $b - a / 2^n$.

The midpoint of this final interval will give us our best approximation for $r$, i.e. $\hat{r} = a_n + b_n / 2$.

Thus we have the error bound $|\hat{r} - r| < b - a / 2^n + 1$. 
Section 1.1

How accurate is Bisection Method?

- Assuming a starting interval \([a, b]\), after \(n\) bisection steps the interval \([a_n, b_n]\) will have length \(\frac{b-a}{2^n}\)

\[\hat{r} = \frac{a_n + b_n}{2}\]

Thus we have the error bound

\[|\hat{r} - r| < \frac{b-a}{2^n + 1}\]
How accurate is Bisection Method?

- Assuming a starting interval \([a, b]\), after \(n\) bisection steps the interval \([a_n, b_n]\) will have length \(\frac{b-a}{2^n}\).

- The midpoint of this final interval will give us our best approximation for \(r\), i.e.

\[
\hat{r} = \frac{a_n + b_n}{2}
\]
How accurate is Bisection Method?

- Assuming a starting interval \([a, b]\), after \(n\) bisection steps the interval \([a_n, b_n]\) will have length \(\frac{b-a}{2^n}\).

- The midpoint of this final interval will give us our best approximation for \(r\), i.e.

\[
\hat{r} = \frac{a_n + b_n}{2}
\]

- Thus we have the error bound

\[
|\hat{r} - r| < \frac{b - a}{2^{n+1}}
\]
1.1

*Definition*: A solution is correct within $p$ decimal places if the error is less than $0.5 \times 10^{-p}$.
Example: p. 29, Exercise 5
Section 1.2

In MATLAB, apply $\cos$ function to any starting number, then apply $\cos$ to the result, then to the new result, etc...

Sequence of numbers converges to 0.7390851332...

Definition - A number $x$ is a fixed point of the function $g$ if $g(x) = x$. NOTE: 0.7390851332 is a fixed point of $\cos(x)$. 
Section 1.2

In MATLAB, apply $\cos$ function to any starting number, then apply $\cos$ to the result, then to the new result, etc...

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Sequence of numbers converges to 0.7390851332...

**Definition** - A number $x$ is a **fixed point** of the function $g$ if $g(x) = x$. NOTE: 0.7390851332 is a fixed point of $\cos(x)$
Section 1.2

Fixed Point Iteration

Solve \( x^3 + x - 1 = 0 \) (Approach 1)

\[ x = 1 - x^3 \]

\[ x = g(x) \]

FAIL
Section 1.2

Fixed Point Iteration

Solve $x^3 + x - 1 = 0$ (Approach 1)
Section 1.2

Fixed Point Iteration

Solve \( x^3 + x - 1 = 0 \) (Approach 1)

\[
x^3 + x - 1 = 0
\]
Section 1.2

Fixed Point Iteration

Solve \( x^3 + x - 1 = 0 \) (Approach 1)

\[
x^3 + x - 1 = 0
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\[
x = 1 - x^3
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Fixed Point Iteration

Solve $x^3 + x - 1 = 0$ (Approach 1)

\[
x^3 + x - 1 = 0
\]
\[
x = 1 - x^3
\]
\[
x = g(x)
\]
Section 1.2

Fixed Point Iteration

Solve $x^3 + x - 1 = 0$ (Approach 1)

\[
x^3 + x - 1 = 0
\]
\[
x = 1 - x^3
\]
\[
x = g(x)
\]
Section 1.2

Fixed Point Iteration

Solve $x^3 + x - 1 = 0$ (Approach 1)

\[ x^3 + x - 1 = 0 \]
\[ x = 1 - x^3 \]
\[ x = g(x) \]

FAIL
Solve $x^3 + x - 1 = 0$ (Approach 2)
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 2)

$x^3 + x - 1 = 0$
Section 1.2

Solve \( x^3 + x - 1 = 0 \) (Approach 2)

\[
x^3 + x - 1 = 0
\]
\[
x^3 = 1 - x
\]
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 2)

\[
\begin{align*}
x^3 + x - 1 &= 0 \\
x^3 &= 1 - x \\
x &= (1 - x)^{\frac{1}{3}}
\end{align*}
\]
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 2)

\[
x^3 + x - 1 = 0
\]

\[
x^3 = 1 - x
\]

\[
x = (1 - x)^{\frac{1}{3}}
\]

\[
x = g(x)
\]
Solve $x^3 + x - 1 = 0$ (Approach 2)

\[
x^3 + x - 1 = 0
\]

\[
x^3 = 1 - x
\]

\[
x = (1 - x)^{\frac{1}{3}}
\]

\[
x = g(x)
\]

SUCCESS!
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 2)

\[
x^3 + x - 1 = 0
\]

\[
x^3 = 1 - x
\]

\[
x = (1 - x)^{\frac{1}{3}}
\]

\[
x = g(x)
\]

SUCCESS!
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 3)
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 3)

$$x^3 + x - 1 = 0$$
Section 1.2

Solve \( x^3 + x - 1 = 0 \) (Approach 3)

\[
\begin{align*}
x^3 + x - 1 &= 0 \\
3x^3 + x &= 1 + 2x^3
\end{align*}
\]
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 3)

\[
x^3 + x - 1 = 0
\]

\[
3x^3 + x = 1 + 2x^3
\]

\[
(3x^2 + 1)x = 1 + 2x^3
\]
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 3)

$$x^3 + x - 1 = 0$$

$$3x^3 + x = 1 + 2x^3$$

$$(3x^2 + 1)x = 1 + 2x^3$$

$$x = \frac{1 + 2x^3}{3x^2 + 1}$$
Solve $x^3 + x - 1 = 0$ (Approach 3)

\[
x^3 + x - 1 = 0
\]
\[
3x^3 + x = 1 + 2x^3
\]
\[
(3x^2 + 1)x = 1 + 2x^3
\]
\[
x = \frac{1 + 2x^3}{3x^2 + 1}
\]
\[
x = g(x)
\]
Section 1.2

Solve \( x^3 + x - 1 = 0 \) (Approach 3)

\[
\begin{align*}
    x^3 + x - 1 &= 0 \\
    3x^3 + x &= 1 + 2x^3 \\
    (3x^2 + 1)x &= 1 + 2x^3 \\
    x &= \frac{1 + 2x^3}{3x^2 + 1} \\
    x &= g(x)
\end{align*}
\]
Section 1.2

Solve $x^3 + x - 1 = 0$ (Approach 3)

\[
x^3 + x - 1 = 0
\]
\[
3x^3 + x = 1 + 2x^3
\]
\[
(3x^2 + 1)x = 1 + 2x^3
\]
\[
x = \frac{1 + 2x^3}{3x^2 + 1}
\]
\[
x = g(x)
\]

SUCCESS!!!

MATH 446/OR 481 Numerical AnalysisLecture 1
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P(\frac{1}{2})$. 

Method 1: 

$$P\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} - 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} - 1 = \frac{5}{4}$$

10 multiplications, 4 additions
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P\left(\frac{1}{2}\right)$.

Method 1:
Assume a polynomial function \( P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1 \). Find \( P\left(\frac{1}{2}\right) \).

Method 1:
\[
P\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} - 3 \cdot \frac{1}{2} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} - 1 = \frac{5}{4}
\]
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P\left(\frac{1}{2}\right)$.

Method 1:
\[ P\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} - 3 \cdot \frac{1}{2} \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} - 1 = \frac{5}{4} \]

10 multiplications, 4 additions
Assume a polynomial function \( P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1 \). Find \( P\left(\frac{1}{2}\right) \).
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P\left(\frac{1}{2}\right)$.

Method 2:
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P\left(\frac{1}{2}\right)$.

Method 2:

\[
\frac{1}{2} \cdot \frac{1}{2} = x \cdot x = x^2
\]
Assume a polynomial function \( P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1 \). Find \( P\left(\frac{1}{2}\right) \).

Method 2:

\[
\frac{1}{2} \cdot \frac{1}{2} = x \cdot x = x^2 \\
x^2 \cdot x = x^3
\]
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P\left(\frac{1}{2}\right)$.

Method 2:

\[
\begin{align*}
\frac{1}{2} \cdot \frac{1}{2} & = x \cdot x = x^2 \\
x^2 \cdot x & = x^3 \\
x^3 \cdot x & = x^4
\end{align*}
\]
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P\left(\frac{1}{2}\right)$.

Method 2:

\[
\begin{align*}
\frac{1}{2} \cdot \frac{1}{2} &= x \cdot x = x^2 \\
x^2 \cdot x &= x^3 \\
x^3 \cdot x &= x^4
\end{align*}
\]

7 multiplications, 4 additions
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P(\frac{1}{2})$. 

Method 3: 

$P(x) = -1 + 5x - 3x^2 + 3x^3 + 2x^4 = -1 + x(5 - 3x + 3x^2 + 2x^3) = -1 + x(5 + x(-3 + x(3 + x(2))))$ 

4 multiplications, 4 additions!!!
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P(\frac{1}{2})$.

Method 3:
Section 0.1

Assume a polynomial function \( P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1 \). Find \( P\left(\frac{1}{2}\right) \).

Method 3:

\[
P(x) = -1 + 5x - 3x^2 + 3x^3 + 2x^4
\]
Section 0.1

Assume a polynomial function \( P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1 \). Find \( P\left(\frac{1}{2}\right) \).

Method 3:

\[
P(x) = -1 + 5x - 3x^2 + 3x^3 + 2x^4 \\
= -1 + x(5 - 3x + 3x^2 + 2x^3)
\]
Assume a polynomial function \( P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1. \)
Find \( P\left(\frac{1}{2}\right)\).

Method 3:

\[
P(x) = -1 + 5x - 3x^2 + 3x^3 + 2x^4 \\
= -1 + x(5 - 3x + 3x^2 + 2x^3) \\
= -1 + x(5 + x(-3 + 3x + 2x^2))
\]
Assume a polynomial function $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$. Find $P\left(\frac{1}{2}\right)$.

Method 3:

$$P(x) = -1 + 5x - 3x^2 + 3x^3 + 2x^4$$
$$= -1 + x(5 - 3x + 3x^2 + 2x^3)$$
$$= -1 + x(5 + x(-3 + 3x + 2x^2))$$
$$= -1 + x(5 + x(-3 + x(3 + x(2))))$$
Assume a polynomial function \( P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1 \). Find \( P\left(\frac{1}{2}\right) \).

Method 3:

\[
P(x) = \frac{1}{2} + 5\frac{1}{2} - 3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 2\left(\frac{1}{2}\right)^4
\]

\[
= \frac{1}{2} + x(5 - 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3)
\]

\[
= \frac{1}{2} + x(5 + x(-3 + 3\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2))
\]

\[
= \frac{1}{2} + x(5 + x(-3 + x(3 + x(2))))
\]

4 multiplications, 4 additions!!!
Homework

- Read 0.1, 1.1, 1.2
- Exercises
  - 0.1: 1, 3
  - 1.1: 1, 3, 5
  - 1.2: 1, 3, 5
- Familiarize yourself with MATLAB