9.1 Limits of Functions.

A. Definition of limit.

1. Definition. We will consider *vector-valued functions*, \( f: D \rightarrow \mathbb{E}^m \), with domain \( D = D_f \subseteq \mathbb{E}^n \). We write
\[
\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1, \ldots, x_n) = (f_1(x_1), f_2(x_2), \ldots, f_m(x_m))
\]
where \( f_i: D \rightarrow \mathbb{R} \) and we usually write
\[
f_i(x) = f_i(x_1, x_2, \ldots, x_n).
\]

2. Definition. Let \( a \) be a limit point (cluster point) of the domain \( D_f \) of a function \( \mathbf{f} \). Then
\[
\lim_{\mathbf{x} \to a} \mathbf{f}(\mathbf{x}) = \mathbf{L}
\]
if for all \( \epsilon > 0 \) there is an \( \delta > 0 \) such that for all \( \mathbf{x} \in D_f \), if \( 0 < \|\mathbf{x} - a\| < \delta \), then \( \|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \epsilon \).

3. Remark. If \( a \) is an isolated point of \( D_f \) then it does not make sense to talk about
\[
\lim_{\mathbf{x} \to a} \mathbf{f}(\mathbf{x}).
\]
Theorem 1. (9.1.1) Suppose that $a$ is a limit point of the domain $D_f$ of the function $f$. Then the following are equivalent

a. $\lim_{x \to a} f(x) = L$.

b. For every sequence $\{x^{(j)}\} \in D_f$, with $x^{(j)} \neq a$ for all $j$, such that $x^{(j)} \to a$, 
   $\lim_{j \to \infty} f(x^{(j)}) = L$.

Proof.
4. Example. Find \( \lim_{(x,y) \to (0,0)} \frac{\sin(x) \sin(y)}{x^2 + y^2} \) or prove it does not exist.

5. Example. Find \( \lim_{(x,y) \to (0,0)} \frac{x^2 + y^4}{x^2 + 2y^4} \) or prove it does not exist.
6. Example. Find \( \lim_{(x,y) \to (0,0)} \frac{x^3 - y^3}{x^2 + y^2} \) or prove it does not exist.