5.5/5.6. The Weierstrass $M$-test and Power Series.

Definition. A sequence of functions $f_k(x)$ defined on a domain $D$ converges to the function $f(x)$ pointwise on $D$ if for each $x \in D$ the numerical sequence $f_k(x)$ converges to $f(x)$. The convergence is uniform if

$$\sup_{x \in D} |f_k(x) - f(x)| \to 0$$

As $k \to \infty$. The series $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise (resp. uniformly) on $D$ if the sequence of partial sums $s_n(x) = \sum_{k=1}^{n} f_k(x)$ converges pointwise (resp. uniformly) on $D$.

Theorem 5.5.2. (Weierstrass $M$-test)
Let $f_k(x)$ be a sequence of functions defined on a domain $D$. Let

$$M_n = \sup_{x \in D} |f_n(x)| = \|f_n\|_{sup} < \infty$$

If $\sum_{n=1}^{\infty} M_n < \infty$ then the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely and uniformly on $D$.

Proof:
Definition (Power Series)

A *power series* centered at \( a \) (or with *base point* \( a \)) is an infinite series of the form

\[
\sum_{k=0}^{\infty} c_k (x - a)^k
\]

where \( c_k \) is a sequence of real coefficients.

**Theorem 5.6.1.** Given a power series \( f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k \), there exists a number \( 0 \leq R \leq \infty \), called the *radius of convergence* with the property that the series converges absolutely on the interval \( |x - a| < R \), absolutely and uniformly on any interval \( [\alpha, \beta] \subseteq (a - R, a + R) \), and diverges for \( |x - a| > R \).

**Proof:**
Theorem. (Abel) Let \( a < b \). If the power series
\[
f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k
\]
converges pointwise on \([a, b]\), then \( f(x) \) is continuous and converges uniformly on \([a, b]\).

Proof.
Theorem. If \( f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k \) has radius of convergence \( R > 0 \), then \( f \) is infinitely differentiable on \((a - R, a + R)\), and \( f^{(n)} \) is given by

\[
f^{(n)}(x) = \sum_{k=n}^{\infty} k(k - 1) \cdots (k - n + 1) c_k (x - a)^{k-n}
\]

which power series also has radius of convergence \( R \).

Proof: