5.1 Series of Constants (continued).

**Definition 5.1.2** (Absolute convergence) Let \( x_n \) be a sequence of numbers. The series \( \sum_{n=1}^{\infty} x_n \) *converges absolutely* if the series \( \sum_{n=1}^{\infty} |x_n| \) converges. In this case, we say that the sequence \( x_n \) is *absolutely summable*. A series that is convergent but not absolutely convergent is called *conditionally convergent*.

**Theorem 5.1.3.** Every absolutely summable sequence is summable.

**Proof:**
**Definition. (Unconditional convergence.)**
A sequence $y_n$ is a *rearrangement* of a sequence $x_n$ if there is a bijection $f: \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $y_n = x_{f(n)}$. A series $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* if every rearrangement $y_n$ of $x_n$ is summable.

**Lemma.** If $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* then for every rearrangement $y_n$,

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n$$

**Proof:**
**Theorem.** A series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if and only if it is unconditionally convergent.

**Proof.**
Example 5.3. (Geometric series)
Given numbers $a$ and $r$ (not necessarily real), the series $\sum_{k=0}^{\infty} ar^k$ is called a geometric series with common ratio $r$.

Lemma. For any $r \neq 1$,

$$s_n = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Proof:
Theorem. The geometric series \( \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \) if \( |r| < 1 \) and diverges otherwise (if \( a \neq 0 \)).

Proof: