5.1 Series of Constants.

Definition 5.1.1 (Convergent series) Let \( x_n \) be a sequence of numbers. The \textit{(infinite) series} \( \sum_{n=1}^{\infty} x_n \) \textit{converges to} \( s \) if the sequence of partial sums, \( s_n = \sum_{k=1}^{n} x_k \) converges to \( s \). In this case, we say that the sequence of terms \( x_n \) is \textit{summable} and that the corresponding series is \textit{convergent}. Otherwise, we say that the series is \textit{divergent}.

Lemma. (Cauchy criterion.) The series \( \sum_{n=1}^{\infty} x_n \) converges if and only if the sequence of partial sums, \( s_n = \sum_{k=1}^{n} x_k \) is Cauchy, that is, given \( \epsilon > 0 \), there is an \( N \) such that if \( n, m \geq N \) then \( |s_n - s_{m-1}| = |\sum_{k=m}^{n} x_k| < \epsilon \).

Theorem 5.1.1 (\( n^{th} \) term test) If \( x_n \) is a summable sequence, then \( x_n \to 0 \).

Proof.
Theorem: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof:
Theorem. (Abel's Formula or Summation by Parts.)

Let $a_n$ and $b_n$ be real-valued sequences and let $A_n = \sum_{k=1}^{n} a_k$. Then for all $n > 1$

$$\sum_{k=1}^{n} a_k b_k = A_n b_n - \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k)$$

Proof:
Theorem. (Dirichlet's Test)
Let $a_n$ and $b_n$ be real-valued sequences and suppose that $s_n = \sum_{k=1}^{n} a_k$ is a bounded sequence and that $b_n \downarrow 0$ (that is, the sequence $b_n$ is decreasing and converges to 0). Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof:
Theorem. (Alternating Series Test)
Suppose that \( x_n \downarrow 0 \). Then the (alternating) series \( \sum_{n=1}^{\infty} (-1)^n x_n \) converges. Moreover, if \( s = \sum_{n=1}^{\infty} (-1)^n x_n \) then \( |s_n - s| \leq x_{n+1} \).

Proof:
Theorem. (Convergence of trigonometric series - Dirichlet)
Suppose that $a_n \downarrow 0$. Then for every $x \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges.

Proof: