Exercise 8.41.

Solution:

(a) For \( n = 1 \) let \( E_k = [k, \infty) \). Then \( E_{k+1} \subseteq E_k \) for all \( k \) but \( \cap_{k=1}^{\infty} E_k = \emptyset \). For \( n > 1 \), you can just set \( E_k = [k, \infty) \times \cdots \times [k, \infty) \) where there are \( n \) terms in the product. Then the same conclusion holds.

(b) As in the hint, let \( x^{(k)} \in E_k \). Since \( E_k \subseteq E_1 \) for all \( k \), \( x^{(k)} \in E_1 \) for all \( k \). Since \( E_1 \) is compact, the Bolzano-Weierstrass Theorem holds, so there is a subsequence \( x^{(k_j)} \) that converges to some \( x \in E^n \). It remains to show that \( x \in \cap_{k=1}^{\infty} E_k \). Let \( m \in \mathbb{N} \). We will show that \( x \in E_m \). Because \( k_j \geq j \), we have that \( k_j \geq m \) as soon as \( j \geq m \). Therefore, if \( j \geq m \), \( x^{(k_j)} \in E_m \). In other words, the tail of the subsequence \( x^{(k_j)} \) is in \( E_m \). Since \( E_m \) is closed, it contains all of its cluster points, and it follows from this that \( x \in E_m \). (If this is not clear to you, you can reason as follows. If \( x \notin E_m \) then because \( x \) is the limit of a sequence of points in \( E_m \), \( x \) satisfies the definition of cluster point of \( E_m \). But this contradicts the assumption that \( E_m \) is closed. Hence \( x \in E_m \).)