1. Let \( x_k \) be absolutely summable. We must show that \( \sum_{k=1}^{\infty} x_k \) converges. We will show that its partial sums are Cauchy. Let \( S_n = \sum_{k=1}^{n} x_k \), and let \( \varepsilon > 0 \). Since \( \sum_{k=1}^{\infty} |x_k| \) converges, there is an \( N \) such that \( n, m \geq N \) implies that \( \sum_{k=m+1}^{n} |x_k| < \varepsilon \). For this \( N \),

\[
|S_n - S_m| = \left| \sum_{k=m+1}^{n} x_k \right| \leq \sum_{k=m+1}^{n} |x_k| < \varepsilon.
\]

Hence \( S_n \) converges and \( x_k \) is summable.

2. Let \( x_k = (-1)^k \). Then \( S_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} (-1)^k \)

\[
= \begin{cases} 
-1 & \text{if } n \text{ odd} \\
0 & \text{if } n \text{ even}
\end{cases}
\]

Hence \( S_n \) is bounded. However, the sequence \( S_n \) does not converge, because always \( |S_{n+1} - S_n| = 1 \), so \( S_n \) is not Cauchy.
3. (⇒) Suppose that $x_n$ is summable. This means that the sequence $s_n$ of partial sums is convergent. But we know that convergent sequences are bounded. Hence $s_n$ is bounded.

(⇐) Suppose that $s_n$ is bounded. We must show it is convergent. Since $x_n \geq 0$ for all $n$, $s_n$ is a non-decreasing sequence.

Let $s_n = \sup s_n$ for all $n$. Since $s_n$ is bounded, then $s = \sup s_n$ exists (by the Completeness Axiom). We will show that $s_n \to s$.

Let $\varepsilon > 0$. Since $s = \sup s_n$, there is an $N$ such that $s_n > s - \varepsilon$, or $s - s_n < \varepsilon$.

Since $s_n \leq s$ for all $n$, $|s - s_n| < \varepsilon$. Since $s_n$ is increasing, $s - s_n$ is decreasing.

So if $n \geq N$, $|s - s_n| < \varepsilon$. Hence $s_n \to s$. 

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4. In order to show $\Sigma$ converges uniformly on $D$ we want to show that given $\varepsilon > 0$ there is an $N$ such that

if $n, m \geq N$ then $\|f_{n} - f_{m}\|_{\sup} = \|\sum_{b=n+1}^{m} f_{b}\|_{\sup} < \varepsilon$.

But since $\{f_{n}\}$ is summable, there is an $N$ such that $n, m \geq N$ implies that

$$\sum_{b=n+1}^{m} f_{b} < \varepsilon.$$ Hence if $n, m \geq N$,

$$\|\sum_{b=m+1}^{n} f_{b}\|_{\sup} = \sum_{b=m+1}^{n} M_{b} < \varepsilon.$$ (Here we have used the fact that $\sup_{x} |f(x) + g(x)| = \sup_{x} |f(x)| + \sup_{x} |g(x)|$)

$\therefore$ $\sum_{b=m+1}^{n} f_{b}$ converges uniformly to $f$ on $D$. 

\[\therefore \]
5. (a) Let \( \mathcal{O} \) be a collection of open sets in \( \mathbb{R}^n \) and let \( \mathcal{O}_0 = \bigcup_{\mathcal{O}} \). We must show \( \mathcal{O}_0 \) is open. Let \( x \in \mathcal{O}_0 \) then for some \( \mathcal{O}_x \), \( x \in \mathcal{O}_x \). Since \( \mathcal{O}_x \) is open there is an \( \varepsilon > 0 \) such that \( B(x, \varepsilon) \subseteq \mathcal{O}_x \). But since \( \mathcal{O}_x \subseteq \mathcal{O}_0 \), \( B(x, \varepsilon) \subseteq \mathcal{O}_0 \) and \( \mathcal{O}_0 \) is open.

(b) Let \( \mathcal{O}_n^{\leq} \) be open sets and let \( \mathcal{O} = \bigcap_{n=1}^{\infty} \mathcal{O}_n \). We must show \( \mathcal{O} \) is open. Let \( x \in \mathcal{O} \) then \( x \in \mathcal{O}_n \) for each \( n \) so there is an \( \varepsilon_n > 0 \) such that \( B(x, \varepsilon_n) \subseteq \mathcal{O}_n \) for each \( n \). Let \( \varepsilon = \min \{ \varepsilon_1, \ldots, \varepsilon_n \} \). We will show \( B(x, \varepsilon) \subseteq \mathcal{O} \). If \( y \in B(x, \varepsilon) \) then since \( \varepsilon \leq \varepsilon_n \) for all \( n \), \( y \in B(x, \varepsilon_n) \subseteq \mathcal{O}_n \). Hence \( y \in \bigcap_{n=1}^{\infty} \mathcal{O}_n \subseteq \mathcal{O} \), and so \( B(x, \varepsilon) \subseteq \mathcal{O} \). Hence \( \mathcal{O} \) is open.