11.1 Definition of the Integral.

A. Integration on the line.

1. **Definition 1.** A *partition* $P$ of an interval $[a, b]$ is a finite set $P = \{x_0, x_1, \ldots, x_n\}$ where
   \[a = x_0 < x_1 < \cdots < x_n = b.\]

   Given a bounded function $f$ defined on $[a, b]$, define
   \[M_k = \sup\{f(x): x \in [x_{k-1}, x_k]\},\]
   and
   \[m_k = \inf\{f(x): x \in [x_{k-1}, x_k]\}.\]

   Define the *upper and lower sums* of $f$ by
   \[U(f, P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})\]
   \[L(f, P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})\]

   If
   \[\sup_P L(f, P) = \inf_P U(f, P).\]

   then $\int_a^b f(x)\,dx$ is this common number and $f$ is said to be *Riemann integrable on* $[a, b]$ or $f \in \mathcal{R}[a, b]$. 
B. Integration on $\mathbb{E}^n$.

1. What replaces the interval $[a, b]$?

**Definition 2.** A *closed rectangular block* (or just a *rectangle*) $B = [a, b] \subseteq \mathbb{E}^n$ is defined by

$$B = [a, b] = [a_1, b_1] \times \cdots \times [a_n, b_n]$$
2. What replaces a partition?

**Definition 3.** Given a rectangle $[a, b]$ in $\mathbb{E}^n$, a *partition* $\mathcal{P}$ of $[a, b]$ is the Cartesian product

$$\mathcal{P} = P_1 \times \cdots \times P_n$$

where each $P_i$ is a partition of $[a_i, b_i]$. A *grid* is a collection of rectangles of the form $I_1 \times \cdots \times I_n$ where each $I_j$ is a subinterval of the partition $P_j$, that is, $I_j = [x_{k-1}^j, x_k^j]$. 
3. We also need to specify the notion of volume.

**Definition 4.** Let $S \subseteq \mathbb{E}^n$ and suppose that $S \subseteq B$ for some rectangle $B$. Let $\mathcal{P}$ be a grid on $B$, and define

$$V(S, \mathcal{P}) = \sum_{B_j \in \mathcal{P} \atop B_j \cap \bar{S} \neq \emptyset} \text{vol}(B_j);$$

$$v(S, \mathcal{P}) = \sum_{B_j \in \mathcal{P} \atop B_j \subseteq S^o} \text{vol}(B_j)$$

where $\bar{S}$ is the closure and $S^o$ the interior of $S$.

The *inner volume* of $S$ is

$$\text{vol}(S) = \sup \{ v(S, \mathcal{P}) : \mathcal{P} \text{ a grid on } B \}$$

and the *outer volume* is

$$\text{Vol}(S) = \inf \{ V(S, \mathcal{P}) : \mathcal{P} \text{ a grid on } B \}$$

If $\text{vol}(S) = \text{Vol}(S)$ then $S$ is called a *Jordan region* and the common value is the *volume* or *Jordan content* of $S$ and is denoted $v(S)$. 
4. **Theorem.** A bounded set $S \subseteq \mathbb{E}^n$ is a Jordan region if and only if $Vol(\partial S) = 0$ where $\partial S = \bar{S} \setminus S^\circ$ is the boundary of $S$.

Such a set is called a *Jordan null set.*
C. Integration over Jordan Regions.

1. **Definition 5.** Let \( S \subseteq \mathbb{E}^n \) be a Jordan region, \( f : S \to \mathbb{E}^1 \) be bounded, and suppose that \( S \subseteq B \) for some rectangle \( B \). Extend \( f \) to a function on \( B \) by letting \( f(x) = 0 \) if \( x \in B \setminus S \).

The *upper sum* of \( f \) with respect to a grid \( \mathcal{P} \) of \( B \) is

\[
U(f, \mathcal{P}) = \sum_{B_j \in \mathcal{P} \atop B_j \cap E \neq \emptyset} M_j \nu(B_j)
\]

where \( M_j = \sup\{f(x) : x \in B_j\} \) and the *lower sum* by

\[
L(f, \mathcal{P}) = \sum_{B_j \in \mathcal{P} \atop B_j \cap E \neq \emptyset} m_j \nu(B_j)
\]

where \( m_j = \inf\{f(x) : x \in B_j\} \).
2. **Definition 6.** The upper and lower integrals of $f$ over $S$ are given by

$$\int_S f = \sup\{L(f,\mathcal{P}) : \mathcal{P} \text{ a grid on } B\}$$

and

$$\int_S f = \inf\{U(f,\mathcal{P}) : \mathcal{P} \text{ a grid on } B\}$$

if $\int_S f = \int_S f$ then the common value is the Riemann integral of $f$ on $S$ and is denoted $\int_S f$.

3. **Remark.**
   
   a. The value of the integral over $S$ is independent of the choice of rectangle $B$ containing $S$.

   b. What is the point of defining the integral this way, i.e., by enclosing $S$ in a rectangle?