10.3 The Chain Rule.

A. The Product Rule (Section 10.2).

1. **Theorem.** Let \( f, g : D \subseteq \mathbb{E}^n \to \mathbb{E}^m \) be differentiable at \( x_0 \in D \). Then
\[
(f \cdot g)'(x_0) = f(x_0)g'(x_0) + g(x_0)f'(x_0)
\]

2. **Remark.** How do we interpret this formula in terms of linear transformations?
3. Proof of Theorem:
B. The Chain Rule.

1. **Theorem.** Suppose that $g: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $f: V \subseteq \mathbb{R}^m \to \mathbb{R}^p$, where $D$ is an open subset of $\mathbb{R}^n$ and $V$ is an open subset of $\mathbb{R}^m$ such that $g(D) \subseteq V$, and that $g'(x_0)$ and $f'(g(x_0))$ both exist at $x_0 \in D$. Then

\[
(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)
\]

2. **Remark.** How do we interpret this theorem in terms of linear transformations?
3. Proof of Theorem.
C. The Mean Value Theorem.

1. **Theorem.** Let \( f: [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Then there is a \( c \in (a, b) \) such that \( f(b) - f(a) = f'(c)(b - a) \).

2. **Remark.** A natural generalization to functions \( f: D \subseteq \mathbb{E}^n \to \mathbb{E}^m \) might be: Suppose that \( f: V \to \mathbb{E}^m \) where \( V \) is a ball in \( \mathbb{E}^n \). Then given \( a, b \in V \) there is a \( c \) on the line segment joining \( a \) and \( b \) such that \( f(b) - f(a) = f'(c)(b - a) \).

3. Note first of all that the dimensions of the matrices work out, but the theorem does not hold.
4. For \( f \) as above, consider the function \( g: \mathbb{R} \to \mathbb{E}^m \) given by \( g(t) = tb + (1 - t)a \). Then look at the function \( f \circ g: \mathbb{R} \to \mathbb{E}^m \). What can we say in this case?
5. **Theorem.** (MVT 1) Let $V \subseteq \mathbb{E}^n$ be open and convex, and let $f: V \to \mathbb{E}^m$ be differentiable on $V$. Let $a, b \in V$ and let $u \in \mathbb{E}^m$ be an arbitrary vector. Then there is a $c$ on the line segment joining $a$ and $b$ such that
\[
u \cdot (f(b) - f(a)) = u \cdot (f'(c)(b - a))\]

6. **Example.** Let $f(x, y) = x(y - 1)$. Then $f(1,1) - f(0,0) = 0$, and $\nabla f(x, y)$ does not vanish on the line segment joining $(0,0)$ and $(1,1)$. 
8. **Theorem.** (MVT 2) Under the hypotheses of the previous theorem, there exist vectors \( c_1, c_2, ..., c_m \in V \subseteq \mathbb{E}^n \) such that

\[
\mathbf{f}(b) - \mathbf{f}(a) = \left[ \frac{\partial f_i}{\partial x_j}(c_j) \right] (b - a)
\]