\[
\frac{f(x+h) - f(x)}{h} \to f'(x)
\]

\[
\frac{f(x+h) - f(x) - f'(x)h}{h} \to 0
\]

\[
f(x+h) - f(x) - mh \to 0 \quad \text{for any } m
\]

\[
\frac{f(x+h) - f(x) - mh}{h} \not\to 0 \quad \text{unless } m = f'(x)
\]
B. The Chain Rule.

1. **Theorem.** Suppose that \( g: D \subseteq \mathbb{E}^n \rightarrow \mathbb{E}^m \) and \( f: V \subseteq \mathbb{E}^m \rightarrow \mathbb{E}^p \), where \( D \) is an open subset of \( \mathbb{E}^n \) and \( V \) is an open subset of \( \mathbb{E}^m \) such that \( g(D) \subseteq V \), and that \( g'(x_0) \) and \( f'(g(x_0)) \) both exist at \( x_0 \in D \). Then
   \[
   (f \circ g)'(x_0) = f'(g(x_0)) g'(x_0)
   \]

2. **Remark.** How do we interpret this theorem in terms of linear transformations?
3. **Proof of Theorem.**

Need to show \( f \circ g (x_0 + h) - f \circ g (x_0) - f'(g(x_0))g'(x_0)h \rightarrow 0 \) faster than \( h \). Note that

\[
\begin{align*}
    f(g(x_0 + h)) - f(g(x_0)) - f'(g(x_0))g'(x_0)h \\
    &= \left[ f(g(x_0 + h)) - f(g(x_0)) - f'(g(x_0))(g(x_0 + h) - g(x_0)) \right] \\
    &\quad + \left[ f'(g(x_0))(g(x_0 + h) - g(x_0) - g(x_0)h) \right] \\
    &= (1) + (2) \\
\end{align*}
\]

Look at (2) first

\[
\frac{\| (2) \|}{\| h \|} = \frac{\| f'(g(x_0))(g(x_0 + h) - g(x_0) - g(x_0)h) \|_{\mathbb{E}^n}}{\| h \|_{\mathbb{E}^n}} \\
\leq \| f'(g(x_0)) \| \frac{\| (g(x_0 + h) - g(x_0) - g(x_0)h) \|_{\mathbb{E}^m}}{\| h \|_{\mathbb{E}^n}} \\
\leq \| f'(g(x_0)) \| \frac{\| h \|_{\mathbb{E}^m}}{\| h \|_{\mathbb{E}^n}} \\
\rightarrow 0 \quad \text{as} \quad \| h \|_{\mathbb{E}^n} \rightarrow 0.
\]
Look at (1). Let \( \varepsilon > 0 \). Need to find \( \delta > 0 \) such that if \( \| \Delta u \| < \delta \) then \( \| \Delta u \| \leq \varepsilon \| \Delta u \| \). We know that
\[
\lim_{\delta \to 0} \frac{\| g'(x+\Delta) - g'(x) \|}{\| \Delta u \|} = 0
\]
So taking there is an \( \eta > 0 \) such that if \( \| \Delta u \| < \eta \) then
\[
\| g'(x+\Delta) - g'(x) \| < \frac{\varepsilon}{1 + \| g'(x) \|}
\]
(just use definition limit with \( \varepsilon = 3 \)). Therefore as long as \( \| \Delta u \| < \eta \) then
\[
\| g'(x+\Delta) - g'(x) \| < \frac{\varepsilon}{1 + \| g'(x) \|} \| \Delta u \|
\]
\[
\leq \| g'(x+\Delta) - g'(x) \| + \| g'(x) \| \| \Delta u \| < (1 + \| g'(x) \|) \| \Delta u \|
\]
For \( \varepsilon > 0 \) above choose \( \delta > 0 \) so that if \( \| \Delta u \| < \delta \) then
\[
\| f(g(x)+\Delta) - f(g(x)) - f'(g(x)) \| < \frac{\varepsilon}{1 + \| g'(x) \|} \| \Delta u \|
\]
Now if \( \| \Delta u \| < \min \left( \eta, \frac{\delta}{1 + \| g'(x) \|} \right) \)
Then \(|g(\hat{x}_0 + \hat{u}) - g(\hat{x}_0)| < \frac{3}{(1 + |g'(\hat{x}_0)|)|\hat{u}|} < \varepsilon\).

So letting \(\hat{z} = g(\hat{x}_0 + \hat{u}) - g(\hat{x}_0)\) we have

\[
|f(g(\hat{x}_0 + \hat{u})) - f(g(\hat{x}_0)) - f'(g(\hat{x}_0))(g(\hat{x}_0 + \hat{u}) - g(\hat{x}_0))| = |f(g(\hat{x}_0 + \hat{z})) - f(g(\hat{x}_0)) - f'(g(\hat{x}_0))\hat{z}| < \frac{\varepsilon}{(1 + |g'(\hat{x}_0)|)|\hat{u}|} = \frac{\varepsilon}{1 + |g'(\hat{x}_0)|} < \varepsilon.
\]
C. The Mean Value Theorem.

1. **Theorem.** Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there is a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

2. **Remark.** A natural generalization to functions $f: D \subseteq \mathbb{E}^n \rightarrow \mathbb{E}^m$ might be: Suppose that $f: V \rightarrow \mathbb{E}^m$ where $V$ is a ball in $\mathbb{E}^n$. Then given $a, b \in V$ there is a $c$ on the line segment joining $a$ and $b$ such that $f(b) - f(a) = f(c)(b - a)$.

3. Note first of all that the dimensions of the matrices work out, but the theorem does not hold.

- **e.g.** $f(t) = (\sin t, \cos t)$
  - $f(2\pi) = f(0)$
  - $f(2\pi) - f(0) = \vec{0}$
  - but no $c \in (0, 2\pi)$ with $f'(c) = \vec{0}$.

- **e.g.** $f(x, y) = x(y - 1)$
  - $\vec{a} = (0, 0)$, $\vec{b} = (1, 1)$
  - Then $f(\vec{b}) - f(\vec{a}) = \vec{0}$ but $f'(x) = \nabla f = (y-1, x) \neq \vec{0}$ on line segment.
4. For $f$ as above, consider the function $g: \mathbb{R} \rightarrow E^m$ given by $g(t) = tb + (1 - t)a$. Then look at the function $f \circ g: \mathbb{R} \rightarrow E^m$. What can we say in this case?

So $f \circ g(t): [0,1] \rightarrow E^n$.

Does MVT work here? \exists \? \ t_0 \in (0,1) \ s.t. \ \frac{f(\bar{b}) - f(\bar{a})}{f(\bar{a})} = (f \circ g)'(t_0) \cdot (\bar{b} - \bar{a}) \ (\?)

But this does not work by previous example.

(idea: Look at each component of $f \circ g$, ie. $(f \circ g)_i: E^1 \rightarrow E^1$ so ordinary MVT holds.)
Since $(f \circ g)_j = \hat{e}_j \cdot (f \circ g)$ we can move generally ask:

Given $\hat{u} \in E^n$ does $M \cup T$ hold for $\hat{u} \cdot (f \circ g)$?
5. **Theorem.** (MVT 1) Let \( V \subseteq \mathbb{E}^n \) be open and convex, and let \( f: V \to \mathbb{E}^m \) be differentiable on \( V \). Let \( a, b \in V \) and let \( u \in \mathbb{E}^m \) be an arbitrary vector. Then there is a \( c \) on the line segment joining \( a \) and \( b \) such that
\[
 u \cdot (f(b) - f(a)) = u \cdot (f'(c)(b - a))
\]

6. **Example.** Let \( f(x, y) = x(y - 1) \). Then \( f(1,1) - f(0,0) = 0 \), and \( \nabla f(x,y) \) does not vanish on the line segment joining \((0,0)\) and \((1,1)\).

\[
f(1,1) - f(0,0) = 0
\]
\[
\nabla f(1,1) = \nabla f(0,0) = 0
\]
\[
f'(c) = \nabla f(c_1, c_2) = \begin{bmatrix} 1 \\ y-1 \end{bmatrix} = [c_2-1, c_1]
\]

Can I find \((c_1, c_2)\) on segment joining \((0,0)\) to \((1,1)\) such that
\[

\begin{bmatrix} c_2 - 1, c_1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = c_2 + c_1 - 1 = 0
\]

Normally \( c \) will depend on \( u \).

\[
\nabla f(1,1) \begin{bmatrix} 1 \end{bmatrix} = 0
\]

Let \( g(t) = t \vec{b} + (1-t) \vec{a} \) for \( t \in [0,1] \). Then \((f \circ g) : [0,1] \to \mathbb{R}^m\) and since \( t \vec{b} + (1-t) \vec{a} \) is in \( U \) for all \( t \), \( f \circ g \) is continuous on \([0,1]\).
Also \( f \circ g \) is differentiable on \((0,1)\) with
\[
(f \circ g)'(t) = \left( f(g(t)) \right)' = f'(g(t))(\vec{b} - \vec{a})
\]

Now let \( F(t) = \tilde{u} \circ (f \circ g)(t) \). Then \( F \) is cont on \([0,1]\) and differentiable on \((0,1)\) with \( F'(t) = \tilde{u} \circ (f \circ g)'(t) = \tilde{u} \circ [f'(g(t))(\vec{b} - \vec{a})] \).

By the MVT there is a \( t_0 \in (0,1) \) such that \( F(1) - F(0) = F'(t_0)(1-0) = F'(t_0) \). That is
\[
\tilde{u} \circ (f(\vec{b}) - f(\vec{a})) = \tilde{u} \circ [f'(g(t_0))(\vec{b} - \vec{a})]
\]
\[
= \tilde{u} \circ (f'(\tilde{c})(\vec{b} - \vec{a})) \quad \overset{t_0=g(t)}{=} = g'(t)
\]